

Stochastic averaging principle for multiscale stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations*

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Abstract

Stochastic averaging principle is a powerful tool for studying qualitative analysis of stochastic dynamical systems with different time-scales. In this paper, we will establish an averaging principle for multiscale stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations with slow and fast time scales. Under suitable conditions, the existence of an averaging equation eliminating the fast variable for this coupled system is proved, and as a consequence, the system can be reduced to a single stochastic complex cubic-quintic Ginzburg-Landau equation with a modified coefficient.

Keywords: Stochastic averaging principle; Stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations; Effective dynamics; slow-fast SPDEs; Strong convergence

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1 Introduction

Linearly coupled complex cubic-quintic Ginzburg-Landau equations

$$\begin{cases} A_t + (-i - \beta)\Delta A + (-i - \gamma)|A|^2 A + (-\mu - i\nu)|A|^4 A - \eta A - i\kappa B = 0 \\ B_t + (-i - \beta)\Delta B + (-i - \gamma)|B|^2 B + (-\mu - i\nu)|B|^4 B - \eta B - i\kappa A = 0 \end{cases}$$

supply a model for ring lasers based on dual-core optical fibers [1, 27, 28, 38, 39], and are noteworthy dynamical systems by themselves [33], the model is also of direct relevance to optics, as it describes a ring laser based on a dual-core fiber with bandwidth-limited gain in both cores (which is modeled by the combination of the linear and quintic losses and cubic gain).

It is said as in [34] that almost all physical systems have a certain hierarchy in which not all components evolve at the same rate, i.e., some of components vary very rapidly, while others change very slowly. In the linearly coupled complex cubic-quintic Ginzburg-Landau equations, the components A, B may vary very differently, namely, one is slow component and the other one is fast component.

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The averaging principle is an important method to extract effective macroscopic dynamic from complex systems with slow component and fast component. In this paper, we will be concerned with the stochastic averaging principle for multiscale stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations

$$(*) \begin{cases} dA^\varepsilon + [(-i - \beta)A_{xx}^\varepsilon + (-i - \gamma)|A^\varepsilon|^2 A^\varepsilon + (-\mu - i\nu)|A^\varepsilon|^4 A^\varepsilon - \eta A^\varepsilon - i\kappa B^\varepsilon]dt = \sigma_1 dW_1 & \text{in } Q \\ dB^\varepsilon + \frac{1}{\varepsilon}[(-i - \beta)B_{xx}^\varepsilon + (-i - \gamma)|B^\varepsilon|^2 B^\varepsilon + (-\mu - i\nu)|B^\varepsilon|^4 B^\varepsilon - \eta B^\varepsilon - i\kappa A^\varepsilon]dt = \frac{1}{\sqrt{\varepsilon}}\sigma_2 dW_2 & \text{in } Q \\ A^\varepsilon(0, t) = 0 = A^\varepsilon(1, t) & \text{in } (0, T) \\ B^\varepsilon(0, t) = 0 = B^\varepsilon(1, t) & \text{in } (0, T) \\ A^\varepsilon(x, 0) = A_0(x) & \text{in } I \\ B^\varepsilon(x, 0) = B_0(x) & \text{in } I, \end{cases}$$

where $T > 0, I = (0, 1), Q = I \times (0, T)$, the stochastic perturbations are of additive type, W_1 and W_2 are mutually independent Wiener processes on a complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, which will be specified later, we denote by \mathbb{E} the expectation with respect to \mathbb{P} . The noise coefficients σ_1 and σ_2 are positive constants and the parameter ε is small and positive, which describes the ratio of time scale between the process A^ε and B^ε . With this time scale the variable A^ε is referred as slow component and B^ε as the fast component.

Many problems in the natural sciences give rise to singularly perturbed systems of stochastic partial differential equations. In the past four decades, singularly perturbed systems have been the focus of extensive research within the framework of averaging methods. The separation of scales is then taken to advantage to derive a reduced equation, which approximates the slow components. Conditions under which the averaging principle can be applied to this kind of system are well known in the classical literature.

Multiscale stochastic partial differential equations(SPDEs) arise as models for various complex systems, such model arises from describing multiscale phenomena in, for example, nonlinear oscillations, material sciences, automatic control, fluids dynamics, chemical kinetics and in other areas leading to mathematical description involving “slow” and “fast” phase variables. The study of the asymptotic behavior of such systems is of great interest. In this respect, the question of how the physical effects at large time scales influence the dynamics of the system is arisen. We focus on this question and show that, under some dissipative conditions on fast variable equation, the complexities effects at large time scales to the asymptotic behavior of the slow component can be omitted or neglected in some sense.

The theory of stochastic averaging principle provides an effective approach for the qualitative analysis of stochastic systems with different time-scales and is relatively mature for stochastic dynamical systems. The theory of averaging principle serves as a tool in study of the qualitative behaviors for complex systems with multiscales, it is essential for describing and understanding the asymptotic behavior of dynamical systems with fast and slow variables. Its basic idea is to approximate the original system by a reduced system. The theory of averaging for deterministic dynamical systems, which was first studied by Bogoliubov [2], has a long and rich history.

The averaging principle in the stochastic differential equations(SDEs) setup was first considered by Khasminskii [29] which proved that an averaging principle holds in weak sense, and has been an active research field on which there is a great deal of literature. Taking into account the generalized and refined results, it is worthy quoting the paper

- convergence in probability: Veretennikov [35, 36], Freidlin and Wentzell [18, 19]; .
- mean-square type convergence: Golec and Ladde [20], Givon and co-workers [21];

- strong convergence: Givon [22] and Golec [23].

We are also referred to [25] and the references therein for recent related work on averaging for stochastic systems in finite dimensional space.

However, there are few results on the averaging principle for stochastic systems in infinite dimensional space, an important contribution in this direction has been given by Cerrai and Freidlin with their paper [7] which appeared in 2009. To the best of our knowledge, this is the first article on the averaging principle for stochastic systems in infinite dimensional space, it presented an averaged principle for slow-fast stochastic reaction-diffusion equations.

Next, we recall the recent results:

- convergence in weak sense (convergence in law): Bréhier [4], Dong and co-workers [12], Fu and co-workers [17];
- convergence in probability: Cerrai and Freidlin [7], Cerrai [8, 9];
- strong convergence: Wang and Roberts [37], Bréhier [4], Fu and co-workers [13, 14, 15, 16], Dong and co-workers [12], Xu and co-workers [40, 41], Bao and co-workers [6], Pei and co-workers [32] .

Almost all the above papers considered the stochastic reaction-diffusion equations or stochastic hyperbolic-parabolic equations, the nonlinear terms are assumed to be Lipschitz continuous and in particular to have linear growth. [12] considers the averaging principle for one dimensional stochastic Burgers equation, this is the first article to deal with highly nonlinear term on this topic.

However, to the best of our knowledge, the averaging principle for the stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations (*) has not been so far solved, a natural question is as follows:

★ *Can we establish the averaging principle for the stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations (*) ? To be more precise, can the slow component A^ε be approximated by the solution \bar{A} which governed by a stochastic cubic-quintic Ginzburg-Landau equation?*

The main object in this paper is to establish an effective approximation for slow process A^ε with respect to the limit $\varepsilon \rightarrow 0$. The main difficulty in this paper is the cubic nonlinear terms and the quintic nonlinear terms.

In this paper, we will take

$$\mu = \gamma = -1, \nu = 1$$

for the sake of simplicity. All the results can be extended without difficulty to the general case.

We define

$$\begin{aligned}\mathcal{L}(A) &= (i + \beta)A_{xx}, \\ \mathcal{F}(A) &= (i + \gamma)|A|^2A = (-1 + i)|A|^2A, \\ \mathcal{G}(A) &= (i\nu + \mu)|A|^4A = (-1 + i)|A|^4A, \\ f(A, B) &= \eta A + i\kappa B, \\ g(A, B) &= \eta B + i\kappa A,\end{aligned}$$

then the linearly coupled complex cubic-quintic Ginzburg-Landau equations (*) becomes

$$\begin{cases} dA^\varepsilon = [\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1 dW_1 & \text{in } Q, \\ dB^\varepsilon = \frac{1}{\varepsilon}[\mathcal{L}(B^\varepsilon) + \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)]dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2 dW_2 & \text{in } Q, \\ A^\varepsilon(0, t) = 0 = A^\varepsilon(1, t) & \text{in } (0, T), \\ B^\varepsilon(0, t) = 0 = B^\varepsilon(1, t) & \text{in } (0, T), \\ A^\varepsilon(x, 0) = A_0(x) & \text{in } I, \\ B^\varepsilon(x, 0) = B_0(x) & \text{in } I. \end{cases} \quad (1.1)$$

1.1 Mathematical setting

We introduce the following mathematical setting:

◊ We denote by $L^2(I)$ the space of all Lebesgue square integrable complex-valued functions on I . The inner product on $L^2(I)$ is

$$(u, v) = \Re \int_I u \bar{v} dx,$$

for any $u, v \in L^2(I)$, where $\bar{\bullet}$ denotes the conjugate of \bullet . The norm on $L^2(I)$ is

$$\|u\| = (u, u)^{\frac{1}{2}},$$

for any $u \in L^2(I)$.

$H^s(I)$ ($s \geq 0$) are the classical Sobolev spaces of complex-valued functions on I . The definition of $H^s(I)$ can be found in [24], the norm on $H^s(I)$ is $\|\cdot\|_{H^s}$.

We set

$$\begin{aligned} X_{p,\tau} &= L^p(\Omega; C([0, \tau]; H^1(I))) \times L^p(\Omega; C([0, \tau]; H^1(I))), \\ Y_\tau &= C([0, \tau]; H^1(I)) \times C([0, \tau]; H^1(I)), \end{aligned}$$

where $p \geq 1, \tau \geq 0$. The norms on $X_{p,\tau}$ and Y_τ are defined as

$$\begin{aligned} \|(u, v)\|_{X_{p,\tau}} &= \|u\|_{L^p(\Omega; C([0, \tau]; H^1(I)))} + \|v\|_{L^p(\Omega; C([0, \tau]; H^1(I)))}, \\ \|(u, v)\|_{Y_\tau} &= \|u\|_{C([0, \tau]; H^1(I))} + \|v\|_{C([0, \tau]; H^1(I))}. \end{aligned}$$

◊ For $i = 1, 2$, let $\{e_{i,k}\}_{k \in \mathbb{N}}$ be eigenvectors of a nonnegative, symmetric operator Q_i with corresponding eigenvalues $\{\lambda_{i,k}\}_{k \in \mathbb{N}}$, such that

$$Q_i e_{i,k} = \lambda_{i,k} e_{i,k}, \quad \lambda_{i,k} > 0, \quad k \in \mathbb{N}.$$

Let W_i be an $L^2(I)$ -valued Q_i -Wiener process with operator Q_i satisfying

$$\text{Tr} Q_i = \sum_{k=1}^{+\infty} \lambda_{i,k} < +\infty, \quad k \in \mathbb{N}$$

and

$$W_i = \sum_{k=1}^{+\infty} \lambda_{i,k}^{\frac{1}{2}} \beta_{i,k}(t) e_{i,k} < +\infty, \quad k \in \mathbb{N} \quad t \geq 0,$$

where $\{\beta_{i,k}\}_{k \in \mathbb{N}} (i = 1, 2)$ are independent real-valued Brownian motions on the probability base $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

We define $\|a\|_{Q_i}^2 \triangleq a^2 \text{Tr} Q_i$, for $i = 1, 2$.

◇ Throughout the paper, the letter C denotes positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

We adopt the following hypothesis (H) throughout this paper:

(H) $\beta > 0, \eta > 0, \kappa \in \mathbb{R}, \alpha \triangleq \frac{\beta\lambda}{2} - \eta > 0$, where $\lambda > 0$ is the smallest constant such that the following inequality holds

$$\|u_x\|^2 \geq \lambda \|u\|^2,$$

where $u \in H_0^1(I)$ or $\int_I u dx = 0$.

1.2 Main results

1.2.1 Well-posedness for (1.1)

Let us explain what we mean by a solution of the multiscale stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations in this article.

Definition 1.1. *If $(A^\varepsilon, B^\varepsilon)$ is an adapted process over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that \mathbb{P} -a.s. the integral equations*

$$\begin{aligned} A^\varepsilon(t) &= S(t)A_0 + \int_0^t S(t-s)(\mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon))(s)ds + \int_0^t S(t-s)\sigma_1 dW_1 \\ B^\varepsilon(t) &= S(\frac{t}{\varepsilon})B_0 + \frac{1}{\varepsilon} \int_0^t S(\frac{t-s}{\varepsilon})(\mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon))(s)ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t S(\frac{t-s}{\varepsilon})\sigma_2 dW_2 \end{aligned}$$

hold true for all $t \in [0, T]$, we say that it is a mild solution for (1.1).

Now, we are in a position to present the first main result in this paper.

Theorem 1.1. *Suppose that the hypothesis (H) holds, for any $\varepsilon \in (0, 1), T > 0$, if $(A_0, B_0) \in H_0^1(I) \times H_0^1(I)$, (1.1) admits a unique mild solution $(A^\varepsilon, B^\varepsilon) \in X_{2,T}$.*

The Banach contraction principle is used as the main tool for proving the existence of mild solutions of SPDE in most of the existing papers. We first apply the fixed point theorem to the corresponding truncated equation and give the local existence of mild solution to (1.1). Then, the energy estimates show that the solution is also global in time.

1.2.2 Stochastic averaging principle for (1.1)

Asymptotical methods play an important role in investigating nonlinear dynamical systems. In particular, the averaging methods provide a powerful tool for simplifying dynamical systems, and obtain approximate solutions to differential equations arising from mechanics, mathematics, physics, control and other areas. In this paper, we use stochastic averaging principle to investigate stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations (1.1).

Now, we are in a position to present the second main result in this paper.

Theorem 1.2. Suppose that the hypothesis (H) holds and $A_0, B_0 \in H_0^1(I)$, $(A^\varepsilon, B^\varepsilon)$ is the solution of (1.1) and \bar{A} is the solution of the effective dynamics equation

$$\begin{cases} d\bar{A} = [\mathcal{L}(\bar{A}) + \mathcal{F}(\bar{A}) + \mathcal{G}(\bar{A}) + \bar{f}(\bar{A})]dt + \sigma_1 dW_1 & \text{in } Q \\ \bar{A}(0, t) = 0 = \bar{A}(1, t) & \text{in } (0, T) \\ \bar{A}(x, 0) = A_0(x) & \text{in } I, \end{cases} \quad (1.2)$$

then for any $T > 0$, any $p > 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{t \in [0, T]} \|A^\varepsilon(t) - \bar{A}(t)\|^{2p} = 0,$$

where

$$\bar{f}(A) = \int_{L^2(I)} f(A, B) \mu^A(dB)$$

and μ^A is an invariant measure for the fast motion with frozen slow component

$$\begin{cases} dB = [\mathcal{L}(B) + \mathcal{F}(B) + \mathcal{G}(B) + g(A, B)]dt + \sigma_2 dW_2 & \text{in } Q \\ B(0, t) = 0 = B(1, t) & \text{in } (0, T) \\ B(x, 0) = B_0(x) & \text{in } I, \end{cases} \quad (1.3)$$

where $A \in L^2(I)$.

Moreover, if $p > \frac{5}{4}$, there exists a positive constant $C(p)$ such that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{2p} \right) \leq C(p) \left(\frac{1}{-\ln \varepsilon} \right)^{\frac{1}{8p}};$$

if $0 < p \leq \frac{5}{4}$, for any $\kappa > 0$, there exists a positive constant $C(\kappa)$ such that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{2p} \right) \leq C(\kappa) \left(\frac{1}{-\ln \varepsilon} \right)^{\frac{2p}{(5+2\kappa)^2}}.$$

Remark 1.1. Our results show that the asymptotic behavior of (1.1) can be characterized by (1.3) with averaged coefficients.

The main strategy for proving Theorem 1.2 is:

We can establish stochastic averaging principle for (1.1), this relies on the moment estimates and the Khasminskii technique already known for SDEs: we introduce an auxiliary process for which the slow component of the fast variable is frozen on small intervals of a subdivision. The introduction of the auxiliary process $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$ provides an intermediate step between the processes A^ε and \bar{A} whose difference we need to estimate.

First, we establish the Hölder continuity of time variable for A^ε which is a crucial step, this relies on the property of semigroup $\{S(t)\}_{t \geq 0}$ and the moment estimates of $(A^\varepsilon, B^\varepsilon)$.

Second, based on this Hölder continuity property, the errors of $A^\varepsilon - \hat{A}^\varepsilon$ and $B^\varepsilon - \hat{B}^\varepsilon$ can be obtained, we will establish convergence of the auxiliary process \hat{B}^ε to the fast solution process B^ε and \hat{A}^ε to the slow solution process A^ε , respectively.

Third, by using the skill of stopping times which was introduced in [12] and the moment estimates of $A^\varepsilon, B^\varepsilon, \hat{A}^\varepsilon, \hat{B}^\varepsilon$, we can establish the errors of $\hat{A}^\varepsilon - \bar{A}$.

Finally, we can establish the errors of $A^\varepsilon - \bar{A}$, and we arrive at Theorem 1.2.

1.3 Main novelties

The main novelties of this paper are twofold:

- The first one is to extend the stochastic averaging principle result to stochastic linearly coupled complex cubic-quintic Ginzburg-Landau equations (1.1).

The previous stochastic averaging principle were established for the nonlinear coupled heat-heat equations and the nonlinear coupled wave-heat equations which are different from the one in this paper.

- The second one is to overcome the no-Lipschitz property of the nonlinear term in (1.1).

The main difficulty in this paper is cubic nonlinear terms $|A|^2 A, |B|^2 B$ and quintic nonlinear terms $|A|^4 A, |B|^4 B$ in (1.1) which are not Lipschitz-continuity, traditional methods can't deal with the difficulty in our problem, thus we need to take new measures. How to treat the cubic nonlinear terms and quintic nonlinear terms is the key of the paper.

We overcome this difficulty by the semigroup approach, stochastic analysis techniques, the skill of stopping times, energy estimate method and refined inequality technique. The crucial tool is Proposition 4.1 which play a vital role in this article, namely the following moment estimates

$$\begin{aligned} & \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|A^\varepsilon(t)\|^{2p}, \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|B^\varepsilon(t)\|^{2p}, \\ & \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|A^\varepsilon(t)\|_{H^1}^{2p}, \sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^T \|B^\varepsilon(t)\|_{H^1}^{2p} dt, \\ & \sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t)\|_{H^1}^{2p}. \end{aligned}$$

These moment estimates will be realized by stochastic tools under suitable assumptions, for example, Itô's formula, Burkholder-Davis-Gundy inequality, energy formula Young inequality and Hölder's inequality, etc.

This paper is organized as follows. In Sec. 2, we present some preliminary results. The fast motion equation (1.3) is study in Sec. 3, we present an exponential ergodicity of a fast equation with the frozen slow component. In Sec. 4, we establish the well-posedness and some a priori estimates for the slow-fast system (1.1) and averaged equation (1.2). In Sec. 5, we derive the stochastic averaging principle in sense of strong convergence for (1.1).

2 Preliminary results

To prove the main theorems some preliminary results will be needed. In this section we gather several technical lemmas.

2.1 The semigroup $\{S(t)\}_{t \geq 0}$ associated to $-\mathcal{L}$

According to [42, P83], the operator $-\mathcal{L}$ is positive, self-adjoint and sectorial on the domain $\mathcal{D}(-\mathcal{L}) = H^2(I) \cap H_0^1(I)$. By spectral theory, we may define the fractional powers $(-\mathcal{L})^\alpha$ of $-\mathcal{L}$ with the domain $\mathcal{D}((-\mathcal{L})^\alpha)$ for any $\alpha \in [0, 1]$. We know that the semigroup $\{S(t)\}_{t \geq 0}$ generated by the operator $-\mathcal{L}$ is analytic on $L^p(I)$ for all $1 \leq p \leq \infty$ and enjoys the following properties

[30]:

$$\begin{aligned}
S(t)(-\mathcal{L})^\alpha &= (-\mathcal{L})^\alpha S(t), & \alpha \geq 0, \\
\|(-\mathcal{L})^\alpha S(t)\varphi\|_{L^p(I)} &\leq Ct^{-\alpha}\|\varphi\|_{L^p(I)}, & \alpha \geq 0, t \geq 0, \\
\|D^j S(t)\varphi\|_{L^q(I)} &\leq Ct^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q}+j)}\|\varphi\|_{L^p(I)}, & q \geq p \geq 1, t \geq 0,
\end{aligned} \tag{2.1}$$

where D^j denotes the j -th order derivative with respect to the spatial variable.

2.2 Some useful inequalities

Lemma 2.1. *If $a, b \in \mathbb{R}$, $p > 0$, it holds that*

$$(|a| + |b|)^p \leq \begin{cases} |a|^p + |b|^p & 0 < p \leq 1, \\ 2^{p-1}(|a|^p + |b|^p) & p > 1. \end{cases}$$

Lemma 2.2. *(Young inequality) Let $a, b \in [0, +\infty)$ and $\varepsilon > 0$, then we have*

$$ab \leq \varepsilon^{-p} \frac{a^p}{p} + \varepsilon^q \frac{b^q}{q},$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.3. *Let $y(t)$ be a nonnegative function, if*

$$y' \leq -ay + f,$$

we have

$$y(t) \leq y(s)e^{-a(t-s)} + \int_s^t e^{-a(t-\tau)} f(\tau) d\tau.$$

2.3 Some useful estimates

The following lemmas are very useful in establishing a priori estimate for the slow-fast system.

Lemma 2.4. *[26, Lemma 7.2] Let A_1 and A_2 be two complex-valued numbers and $\sigma \geq \frac{1}{2}$. Then the following inequality is fulfilled*

$$||A_1|^{2\sigma} A_1 - |A_2|^{2\sigma} A_2| \leq (4\sigma - 1)(|A_1|^{2\sigma} + |A_2|^{2\sigma})|A_1 - A_2|.$$

Remark 2.1. *The same result can be found in [5, P8].*

Lemma 2.5. *[26, Lemma 7.3] Let A_1 and A_2 be two complex-valued numbers and $\sigma > 0$. Then the following inequality is fulfilled*

$$\Re\{(A_1 - A_2)(|A_1|^{2\sigma} A_1 - |A_2|^{2\sigma} A_2)\} \geq 0.$$

Thus we have

Corollary 2.1. *For any $A_1, A_2 \in \mathbb{C}$, we have*

$$\begin{aligned}
(A_1 - A_2, \mathcal{F}(A_1) - \mathcal{F}(A_2)) &\leq 0, \\
(A_1 - A_2, \mathcal{G}(A_1) - \mathcal{G}(A_2)) &\leq 0.
\end{aligned}$$

The following lemma is very useful in establishing a priori estimate for the slow-fast system.

Lemma 2.6. [42, Lemma 2.6] *If $\sigma > 0, |\alpha| < \frac{\sqrt{2\sigma+1}}{\sigma}$, there exists a positive constant λ_α such that*

$$(-A_{xx}, (-1 + \alpha i)|A|^{2\sigma}A) + \lambda_\alpha \int_I |A|^{2\sigma} |A_x|^2 dx \leq 0.$$

In particularity, we have

$$(-A_{xx}, (-1 + \alpha i)|A|^{2\sigma}A) \leq 0.$$

Remark 2.2. *The same results can be found in [26, Lemma 7.4].*

3 The fast motion equation (1.3)

First, we consider the stochastic Ginzburg-Landau equation, the solution of (1.3) will be denoted by B^{A, B_0} .

We could have the following property for the solution of (1.3):

Lemma 3.1. *For $A \in L^2(I)$, let $B^{A, X}$ be the solution of*

$$\begin{cases} dB = [\mathcal{L}(B) + \mathcal{F}(B) + \mathcal{G}(B) + \eta B + i\kappa A]dt + \sigma_2 dW_2 & \text{in } I \times (0, +\infty) \\ B(0, t) = 0 = B(1, t) & \text{in } (0, +\infty) \\ B(x, 0) = X(x) & \text{in } I. \end{cases} \quad (3.1)$$

1) *There exists a positive constant C such that $B^{A, X}$ satisfies:*

$$\begin{aligned} \mathbb{E}\|B^{A, X}(t)\|^2 &\leq e^{-2\alpha t}\|X\|^2 + C(\|A\|^2 + 1), \\ \mathbb{E}\|B^{A, X}(t) - B^{A, Y}(t)\|^2 &\leq \|X - Y\|^2 e^{-2\alpha t}, \end{aligned} \quad (3.2)$$

for $t \geq 0$.

2) *There is unique invariant measure μ^A for the Markov semigroup P_t^A associated with the system (3.1) in $L^2(I)$. Moreover, we have*

$$\int_{L^2(I)} \|z\|^2 \mu^A(dz) \leq C(1 + \|A\|^2).$$

3) *There exists a positive constant C such that $B^{A, X}$ satisfies:*

$$\|\mathbb{E}f(A, B^{A, X}) - \bar{f}(A)\|^2 \leq C(1 + \|X\|^2 + \|A\|^2)e^{-2\alpha t}$$

for $t \geq 0$.

Proof. 1) • By applying the generalized Itô formula with $\frac{1}{2}\|B^{A, X}\|^2$, we can obtain that

$$\begin{aligned} \frac{1}{2}\|B^{A, X}\|^2 &= \frac{1}{2}\|X\|^2 + \int_0^t (B^{A, X}, \mathcal{L}B^{A, X} + \mathcal{F}(B^{A, X}) + \mathcal{G}(B^{A, X}) + \eta B^{A, X} + i\kappa A)ds \\ &\quad + \int_0^t (B^{A, X}, \sigma_2 dW_2) + \frac{1}{2} \int_0^t \|\sigma_2\|_{Q_2}^2 ds \\ &= \frac{1}{2}\|X\|^2 - \beta \int_0^t \|B_x^{A, X}\|^2 ds + \eta \int_0^t \|B^{A, X}\|^2 ds + \int_0^t (B^{A, X}, i\kappa A)ds \\ &\quad + \int_0^t (B^{A, X}, \mathcal{F}(B^{A, X}) + \mathcal{G}(B^{A, X}))ds + \int_0^t (B^{A, X}, \sigma_2 dW_2) + \frac{1}{2} \int_0^t \|\sigma_2\|_{Q_2}^2 ds. \end{aligned}$$

Taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned}\mathbb{E}\|B^{A,X}\|^2 &= \|X\|^2 - 2\beta \int_0^t \mathbb{E}\|B_x^{A,X}\|^2 ds + 2\eta \int_0^t \mathbb{E}\|B^{A,X}\|^2 ds + 2 \int_0^t \mathbb{E}(B^{A,X}, i\kappa A) ds \\ &\quad + 2 \int_0^t \mathbb{E}(B^{A,X}, \mathcal{F}(B^{A,X}) + \mathcal{G}(B^{A,X})) ds + \int_0^t \|\sigma_2\|_{Q_2}^2 ds,\end{aligned}$$

namely,

$$\begin{aligned}\frac{d}{dt} \mathbb{E}\|B^{A,X}\|^2 &= -2\beta \mathbb{E}\|B_x^{A,X}\|^2 + 2\eta \mathbb{E}\|B^{A,X}\|^2 + 2\mathbb{E}(B^{A,X}, i\kappa A) + 2\mathbb{E}(B^{A,X}, \mathcal{F}(B^{A,X}) + \mathcal{G}(B^{A,X})) + \|\sigma_1\|_{Q_1}^2.\end{aligned}$$

According to Corollary 2.1, we have

$$(B^{A,X}, \mathcal{F}(B^{A,X}) + \mathcal{G}(B^{A,X})) \leq 0,$$

thus,

$$\begin{aligned}\frac{d}{dt} \mathbb{E}\|B^{A,X}\|^2 &\leq -2\beta \mathbb{E}\|B_x^{A,X}\|^2 + 2\eta \mathbb{E}\|B^{A,X}\|^2 + \beta \lambda \mathbb{E}\|B^{A,X}\|^2 + C(\lambda, \beta, \kappa) \|A\|^2 + \|\sigma_1\|_{Q_1}^2 \\ &\leq -2\beta \mathbb{E}\|B_x^{A,X}\|^2 + 2\eta \mathbb{E}\|B^{A,X}\|^2 + \beta \mathbb{E}\|B_x^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|_{Q_1}^2 \\ &= -\beta \mathbb{E}\|B_x^{A,X}\|^2 + 2\eta \mathbb{E}\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|_{Q_1}^2 \\ &\leq -\beta \lambda \mathbb{E}\|B^{A,X}\|^2 + 2\eta \mathbb{E}\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|_{Q_1}^2 \\ &= -(\beta \lambda - 2\eta) \mathbb{E}\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|_{Q_1}^2 \\ &= -2\alpha \mathbb{E}\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|_{Q_1}^2.\end{aligned}$$

Hence, by applying Lemma 2.3 with $\mathbb{E}\|B^{A,X}\|^2$, we have

$$\mathbb{E}\|B^{A,X}(t)\|^2 \leq e^{-2\alpha t} \|X\|^2 + C(\|A\|^2 + 1).$$

• It is easy to see

$$\begin{cases} d(B^{A,X} - B^{A,Y}) = [\mathcal{L}(B^{A,X} - B^{A,Y}) + \mathcal{F}(B^{A,X}) - \mathcal{F}(B^{A,Y}) \\ \quad + \mathcal{G}(B^{A,X}) - \mathcal{G}(B^{A,Y}) + \eta(B^{A,X} - B^{A,Y})] dt & \text{in } Q \\ (B^{A,X} - B^{A,Y})(0, t) = 0 = (B^{A,X} - B^{A,Y})(1, t) & \text{in } (0, T) \\ (B^{A,X} - B^{A,Y})(x, 0) = X - Y & \text{in } I, \end{cases}$$

thus, it follows from the energy method that

$$\begin{aligned}\frac{1}{2} \|B^{A,X} - B^{A,Y}\|^2 &= \frac{1}{2} \|X - Y\|^2 + \int_0^t (B^{A,X} - B^{A,Y}, \mathcal{L}(B^{A,X} - B^{A,Y}) + \mathcal{F}(B^{A,X}) - \mathcal{F}(B^{A,Y}) \\ &\quad + \mathcal{G}(B^{A,X}) - \mathcal{G}(B^{A,Y}) + \eta(B^{A,X} - B^{A,Y})) ds \\ &= \frac{1}{2} \|X - Y\|^2 - \beta \int_0^t \|(B^{A,X} - B^{A,Y})_x\|^2 ds + \eta \int_0^t \|B^{A,X} - B^{A,Y}\|^2 ds \\ &\quad + \int_0^t (B^{A,X} - B^{A,Y}, \mathcal{F}(B^{A,X}) - \mathcal{F}(B^{A,Y}) + \mathcal{G}(B^{A,X}) - \mathcal{G}(B^{A,Y})) ds,\end{aligned}$$

namely,

$$\begin{aligned} & \frac{d}{dt} \|B^{A,X} - B^{A,Y}\|^2 \\ &= -2\beta \|(B^{A,X} - B^{A,Y})_x\|^2 + 2\eta \|B^{A,X} - B^{A,Y}\|^2 \\ & \quad + 2(B^{A,X} - B^{A,Y}, \mathcal{F}(B^{A,X}) - \mathcal{F}(B^{A,Y}) + \mathcal{G}(B^{A,X}) - \mathcal{G}(B^{A,Y})). \end{aligned}$$

It follows from Lemma 2.1, we have

$$\begin{aligned} (B^{A,X} - B^{A,Y}, \mathcal{F}(B^{A,X}) - \mathcal{F}(B^{A,Y})) &\leq 0, \\ (B^{A,X} - B^{A,Y}, \mathcal{G}(B^{A,X}) - \mathcal{G}(B^{A,Y})) &\leq 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \frac{d}{dt} \|B^{A,X} - B^{A,Y}\|^2 \\ & \leq -2\beta \|(B^{A,X} - B^{A,Y})_x\|^2 + 2\eta \|B^{A,X} - B^{A,Y}\|^2 \\ & \leq -(\beta\lambda - 2\eta) \|B^{A,X} - B^{A,Y}\|^2 \\ & = -2\alpha \|B^{A,X} - B^{A,Y}\|^2, \end{aligned}$$

this yields

$$\|B^{A,X} - B^{A,Y}\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.$$

Thus, we have

$$\mathbb{E} \|B^{A,X} - B^{A,Y}\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.$$

2) (3.2) imply for any $A \in L^2(I)$ that there is unique invariant measure μ^A for the Markov semigroup P_t^A associated with the system (3.1) in $L^2(I)$ such that

$$\int_{L^2(I)} P_t^A \varphi d\mu^A = \int_{L^2(I)} \varphi d\mu^A, \quad t \geq 0$$

for any $\varphi \in B_b(L^2(I))$ the space of bounded functions on $L^2(I)$.

Then by repeating the standard argument as in [9, Proposition 4.2] and [7, Lemma 3.4], the invariant measure satisfies

$$\int_{L^2(I)} \|z\|^2 \mu^A(dz) \leq C(1 + \|A\|^2).$$

3) According to the invariant property of μ^A , (2) and (3.2), we have

$$\begin{aligned} & \|\mathbb{E} f(A, B^{A,X}) - \bar{f}(A)\|^2 \\ &= \|\mathbb{E} f(A, B^{A,X}) - \int_{L^2(I)} f(A, Y) \mu^A(dY)\|^2 \\ &= \|\mathbb{E} f(A, B^{A,X}) - \mathbb{E} \int_{L^2(I)} f(A, B^{A,Y}) \mu^A(dY)\|^2 \\ &= \|\int_{L^2(I)} \mathbb{E}[f(A, B^{A,X}) - f(A, B^{A,Y})] \mu^A(dY)\|^2 \\ &\leq C \int_{L^2(I)} \mathbb{E} \|B^{A,X} - B^{A,Y}\|^2 \mu^A(dY) \\ &\leq C \int_{L^2(I)} \|X - Y\|^2 e^{-2\alpha t} \mu^A(dY) \\ &\leq C(1 + \|X\|^2 + \|A\|^2) e^{-2\alpha t}. \end{aligned}$$

□

4 Well-posedness and a priori estimate for the slow-fast system (1.1) and averaged equation (1.2)

We first establish the well-posedness for the slow-fast system (1.1).

Since nonlinear terms $\mathcal{F}(A), \mathcal{G}(A), \mathcal{F}(B)$ and $\mathcal{G}(B)$ are not Lipschitz continuous, we will use a truncation argument which will lead to a local existence result. Then via some a priori estimates we obtain that the solution is also global.

4.1 Well-posedness and a priori estimate for the slow-fast system (1.1)

The proof of well-posedness for the slow-fast system (1.1) is divided into several steps.

4.1.1 Local existence

We can establish the local well-posedness for the slow-fast system (1.1) in $X_{p,T}(p \geq 1)$.

Lemma 4.1. *For any $(A_0, B_0) \in H_0^1(I) \times H_0^1(I)$ and $p \geq 1$, $\varepsilon \in (0, 1)$ (1.1) admits a unique mild solution $(A^\varepsilon, B^\varepsilon) \in X_{p,\tau_\infty}$, where τ_∞ is stopping time for p . Moreover, if $\tau_\infty < +\infty$, then \mathbb{P} -a.s.*

$$\limsup_{t \rightarrow \tau_\infty} \|(A^\varepsilon, B^\varepsilon)\|_{Y_t} = +\infty.$$

Proof. Inspired from [24], let $\rho \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\rho(r) = 1$ for $r \in [0, 1]$ and $\rho(r) = 0$ for $r \geq 2$. For any $R > 0, y \in X_{p,t}$ and $t \in [0, T]$, we set

$$\rho_R(y)(t) = \rho\left(\frac{\|y\|_{C([0,t];H^1(I))}}{R}\right).$$

The truncated equation corresponding to (1.1) is the following stochastic partial differential equation:

$$\begin{cases} dA^\varepsilon = [\mathcal{L}(A^\varepsilon) + \rho_R(A^\varepsilon)\mathcal{F}(A^\varepsilon) + \rho_R(A^\varepsilon)\mathcal{G}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1 dW_1 & \text{in } Q, \\ dB^\varepsilon = \frac{1}{\varepsilon}[\mathcal{L}(B^\varepsilon) + \rho_R(B^\varepsilon)\mathcal{F}(B^\varepsilon) + \rho_R(B^\varepsilon)\mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)]dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2 dW_2 & \text{in } Q, \\ A^\varepsilon(0, t) = 0 = A^\varepsilon(1, t) & \text{in } (0, T), \\ B^\varepsilon(0, t) = 0 = B^\varepsilon(1, t) & \text{in } (0, T), \\ A^\varepsilon(x, 0) = A_0(x) & \text{in } I, \\ B^\varepsilon(x, 0) = B_0(x) & \text{in } I. \end{cases}$$

In the proof of Lemma 4.1, we will take

$$\varepsilon = 1$$

for the sake of simplicity. All the results can be extended without difficulty to the general case. Thus, we consider the following system

$$\begin{cases} dA = [\mathcal{L}(A) + \rho_R(A)\mathcal{F}(A) + \rho_R(A)\mathcal{G}(A) + f(A, B)]dt + \sigma_1 dW_1 & \text{in } Q, \\ dB = [\mathcal{L}(B) + \rho_R(B)\mathcal{F}(B) + \rho_R(B)\mathcal{G}(B) + g(A, B)]dt + \sigma_2 dW_2 & \text{in } Q, \\ A(0, t) = 0 = A(1, t) & \text{in } (0, T), \\ B(0, t) = 0 = B(1, t) & \text{in } (0, T), \\ A(x, 0) = A_0(x) & \text{in } I, \\ B(x, 0) = B_0(x) & \text{in } I. \end{cases}$$

We define

$$\begin{aligned} & \Phi_R(A, B) \\ &= \begin{pmatrix} \Phi_R^1(A, B) \\ \Phi_R^2(A, B) \end{pmatrix} \\ &= \begin{pmatrix} S(t)A_0 + \int_0^t S(t-s)(\rho_R(A)\mathcal{F}(A) + \rho_R(A)\mathcal{G}(A) + f(A, B))(s)ds + \int_0^t S(t-s)\sigma_1 dW_1 \\ S(t)B_0 + \int_0^t S(t-s)(\rho_R(B)\mathcal{F}(B) + \rho_R(B)\mathcal{G}(B) + g(A, B))(s)ds + \int_0^t S(t-s)\sigma_2 dW_2 \end{pmatrix}. \end{aligned}$$

- It is easy to see the operator $\Phi_R(A, B)$ maps X_{p, T_0} into itself.
- The estimates of

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^1(A_1, B_1) - \Phi_R^1(A_2, B_2))(t)\|_{H^1}^p, \\ & \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^2(A_1, B_1) - \Phi_R^2(A_2, B_2))(t)\|_{H^1}^p. \end{aligned}$$

Indeed, due to [42, P84], we have

$$\|\rho_R(A_1)|A_1|^{2\sigma}A_1 - \rho_R(A_2)|A_2|^{2\sigma}A_2\| \leq CR^{2\sigma}\|A_1 - A_2\|_{H^1}.$$

By taking $p = q = 2, j = 1$ in the third inequality of (2.1), we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T_0} \left\| \int_0^t S(t-s)(\rho_R(A_1)\mathcal{F}(A_1) - \rho_R(A_2)\mathcal{F}(A_2))(s)ds \right\|_{H^1}^p \\ & \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|(\rho_R(A_1)\mathcal{F}(A_1) - \rho_R(A_2)\mathcal{F}(A_2))(s)\|_{H^1} ds \right)^p \\ & \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} R^2 \|(A_1 - A_2)(s)\|_{H^1} ds \right)^p \\ & \leq CR^{2p} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} ds \right)^p \mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p \\ & \leq CR^{2p} T_0^{\frac{p}{2}} \mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p, \end{aligned} \tag{4.1}$$

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T_0} \left\| \int_0^t S(t-s)(\rho_R(A_1)\mathcal{G}(A_1) - \rho_R(A_2)\mathcal{G}(A_2))(s)ds \right\|_{H^1}^p \\ & \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|(\rho_R(A_1)\mathcal{G}(A_1) - \rho_R(A_2)\mathcal{G}(A_2))(s)\|_{H^1} ds \right)^p \\ & \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} R^4 \|(A_1 - A_2)(s)\|_{H^1} ds \right)^p \\ & \leq CR^{4p} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} ds \right)^p \mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p \\ & \leq CR^{4p} T_0^{\frac{p}{2}} \mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T_0} \left\| \int_0^t S(t-s)(f(A_1, B_1) - f(A_2, B_2))(s) ds \right\|_{H^1}^p \\
& \leq \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t \|S(t-s)(f(A_1, B_1) - f(A_2, B_2))(s)\|_{H^1} ds \right)^p \\
& \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|(f(A_1, B_1) - f(A_2, B_2))(s)\| ds \right)^p \\
& \leq C \mathbb{E} \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} (\|(A_1 - A_2)(s)\| + \|(B_1 - B_2)(s)\|) ds \right)^p \\
& \leq C \sup_{0 \leq t \leq T_0} \left(\int_0^t (t-s)^{-\frac{1}{2}} ds \right)^p (\mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|^p) \\
& \leq C T_0^{\frac{p}{2}} (\mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|^p).
\end{aligned} \tag{4.3}$$

Finally, collecting the above estimates (4.1)-(4.3), we get

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^1(A_1, B_1) - \Phi_R^1(A_2, B_2))(t)\|_{H^1}^p \\
& \leq C(R^{2p}T_0^{\frac{p}{2}} + R^{4p}T_0^{\frac{p}{2}} + T_0^{\frac{p}{2}})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|_{H^1}^p).
\end{aligned} \tag{4.4}$$

By the same method, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^2(A_1, B_1) - \Phi_R^2(A_2, B_2))(t)\|_{H^1}^p \\
& \leq C(R^{2p}T_0^{\frac{p}{2}} + R^{4p}T_0^{\frac{p}{2}} + T_0^{\frac{p}{2}})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|_{H^1}^p).
\end{aligned} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^1(A_1, B_1) - \Phi_R^1(A_2, B_2))(t)\|_{H^1}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(\Phi_R^2(A_1, B_1) - \Phi_R^2(A_2, B_2))(t)\|_{H^1}^p \\
& \leq C(R^{2p}T_0^{\frac{p}{2}} + R^{4p}T_0^{\frac{p}{2}} + T_0^{\frac{p}{2}})(\mathbb{E} \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|_{H^1}^p),
\end{aligned}$$

namely, we have

$$\begin{aligned}
& \|\Phi_R(A_1, B_1) - \Phi_R(A_2, B_2)\|_{X_{p,T_0}} \\
& \leq C(R^2T_0^{\frac{1}{2}} + R^4T_0^{\frac{1}{2}} + T_0^{\frac{1}{2}})\|(A_1, B_1) - (A_2, B_2)\|_{X_{p,T_0}}.
\end{aligned} \tag{4.6}$$

- For a sufficiently small T_0 , is $\Phi_R(A, B)$ a contraction mapping on X_{p,T_0} .

Hence, by applying the Banach contraction principle, $\Phi_R(A, B)$ has a unique fixed point in X_{p,T_0} , which is the unique local solution to (1.1) on the interval $[0, T_0]$. Since T_0 does not depend on the initial value (A_0, B_0) , this solution may be extended to the whole interval $[0, T]$.

We denote by (A_R, B_R) this unique mild solution and let

$$\tau_R = \inf\{t \geq 0 : \|(A_R, B_R)\|_{X_{p,t}} \geq R\},$$

with the usual convention that $\inf \emptyset = \infty$.

Since $R_1 \leq R_2$, $\tau_{R_1} \leq \tau_{R_2}$, we can put $\tau_\infty = \lim_{R \rightarrow +\infty} \tau_R$. We define a local solution to (1.1) as follows

$$\begin{aligned} A(t) &= A_R(t), \quad \forall t \in [0, \tau_R], \\ B(t) &= B_R(t), \quad \forall t \in [0, \tau_R]. \end{aligned}$$

Indeed, for any $t \in [0, \tau_{R_1} \wedge \tau_{R_2}]$

$$\begin{aligned} & A_{R_1}(t) - A_{R_2}(t) \\ &= \int_0^t S(t-s) (\rho_{R_1}(A_{R_1})\mathcal{F}(A_{R_1}) - \rho_{R_2}(A_{R_2})\mathcal{F}(A_{R_2}) + \rho_{R_1}(A_{R_1})\mathcal{G}(A_{R_1}) - \rho_{R_2}(A_{R_2})\mathcal{G}(A_{R_2}) \\ &\quad + f(A_{R_1}, B_{R_1}) - f(A_{R_2}, B_{R_2}))(s) ds, \\ & B_{R_1}(t) - B_{R_2}(t) \\ &= \int_0^t S(t-s) (\rho_{R_1}(B_{R_1})\mathcal{F}(B_{R_1}) - \rho_{R_2}(B_{R_2})\mathcal{F}(B_{R_2}) + \rho_{R_1}(B_{R_1})\mathcal{G}(B_{R_1}) - \rho_{R_2}(B_{R_2})\mathcal{G}(B_{R_2}) \\ &\quad + g(A_{R_1}, B_{R_1}) - g(A_{R_2}, B_{R_2}))(s) ds. \end{aligned}$$

Proceeding as in the proof of (4.6), we can obtain

$$\begin{aligned} & \| (A_{R_1}, B_{R_1}) - (A_{R_2}, B_{R_2}) \|_{X_{p,t}} \\ & \leq C(t) \| (A_{R_1}, B_{R_1}) - (A_{R_2}, B_{R_2}) \|_{X_{p,t}}, \end{aligned}$$

where $C(t)$ is a monotonically increasing function and $C(0) = 0$. If we take t sufficiently small, we can obtain

$$\begin{aligned} A_{R_1}(t) &= A_{R_2}(t), \\ B_{R_1}(t) &= B_{R_2}(t). \end{aligned}$$

Repeating the same argument for the interval $[t, 2t]$ and so on yields

$$\begin{aligned} A_{R_1}(t) &= B_{R_2}(t), \\ A_{R_1}(t) &= B_{R_2}(t), \end{aligned}$$

for the whole interval $[0, \tau]$. According to this, we can know that the above definition of local solution to (1.1) is well defined.

If $\tau_\infty < +\infty$, the definition of (A, B) yields P -a.s.

$$\lim_{t \rightarrow \tau_\infty} \|(A, B)\|_{X_{p,t}} = +\infty,$$

which shows that (A, B) is a unique local solution to (1.1) on the interval $[0, \tau_\infty)$.

This completes the proof of Lemma 4.1. □

4.1.2 Some energy inequalities for the slow-fast system (1.1)

Next, we will exploit some energy inequalities for the slow-fast system (1.1).

Lemma 4.2. Let $\xi = \inf\{\tau_\infty, T\}$. If $A_0, B_0 \in H_0^1(I)$, for $\varepsilon \in (0, 1)$, $(A^\varepsilon, B^\varepsilon)$ is the unique solution to (1.1), then there exists a constant C such that the solutions $(A^\varepsilon, B^\varepsilon)$ satisfy

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{t \in [0, \xi]} \|A^\varepsilon(t)\|_{H^1}^2 &\leq C, \\ \mathbb{E} \sup_{t \in [0, \xi]} \|B^\varepsilon(t)\|_{H^1}^2 &\leq \frac{C}{\varepsilon}, \\ \sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 dt &\leq C, \\ \sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt &\leq C, \end{aligned}$$

where C is dependent of T, A_0, B_0 but independent of $\varepsilon \in (0, 1)$.

Proof. The proof of Lemma 4.2 is divided into several steps. Here, the method of the proof is inspired from [12, 13, 14, 15, 16, 42].

- The estimates of $\sup_{t \in [0, \xi]} \mathbb{E} \|A^\varepsilon\|_{H^1}^2$ and $\sup_{t \in [0, \xi]} \mathbb{E} \|B^\varepsilon\|_{H^1}^2$.

★ Indeed, we apply the generalized Itô formula (see [42, 10, 11, 31]) with $\|A_x^\varepsilon\|^2$ and obtain that

$$\begin{aligned} d\|A_x^\varepsilon\|^2 &= 2(-A_{xx}^\varepsilon, [\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1 dW_1) + \|\sigma_1\|_{Q_1}^2 dt \\ &= -2\beta \|A_{xx}^\varepsilon\|^2 dt + 2\eta \|A_x^\varepsilon\|^2 dt + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon)dt \\ &\quad + 2(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon))dt + 2(-A_{xx}^\varepsilon, \sigma_1 dW_1) + \|\sigma_1\|_{Q_1}^2 dt \\ &= (-2\beta \|A_{xx}^\varepsilon\|^2 + 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + 2(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) + \|\sigma_1\|_{Q_1}^2)dt \\ &\quad + 2(-A_{xx}^\varepsilon, \sigma_1 dW_1), \end{aligned} \tag{4.7}$$

namely, we have

$$\begin{aligned} \|A_x^\varepsilon\|^2 &= \|A_{0x}\|^2 - 2\beta \int_0^t \|A_{xx}^\varepsilon\|^2 ds + 2\eta \int_0^t \|A_x^\varepsilon\|^2 ds + 2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon) ds \\ &\quad + 2 \int_0^t (-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) ds + 2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds, \end{aligned} \tag{4.8}$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned} \mathbb{E} \|A_x^\varepsilon\|^2 &= \mathbb{E} \|A_{0x}\|^2 - 2\beta \mathbb{E} \int_0^t \|A_{xx}^\varepsilon\|^2 ds + 2\eta \mathbb{E} \int_0^t \|A_x^\varepsilon\|^2 ds + 2\mathbb{E} \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon) ds \\ &\quad + 2\mathbb{E} \int_0^t (-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) ds + \mathbb{E} \int_0^t \|\sigma_1\|_{Q_1}^2 ds. \end{aligned}$$

It is easy to see

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|A_x^\varepsilon\|^2 &= -2\beta \mathbb{E} \|A_{xx}^\varepsilon\|^2 + 2\eta \mathbb{E} \|A_x^\varepsilon\|^2 + 2\mathbb{E} (A_x^\varepsilon, i\kappa B_x^\varepsilon) \\ &\quad + 2\mathbb{E} (-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) + \mathbb{E} \|\sigma_1\|_{Q_1}^2. \end{aligned}$$

According to Lemma 2.6, we have

$$(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) \leq 0,$$

thus, it holds that

$$\begin{aligned} & \frac{d}{dt} \mathbb{E} \|A_x^\varepsilon\|^2 \\ & \leq -2\beta \mathbb{E} \|A_{xx}^\varepsilon\|^2 + 2\eta \mathbb{E} \|A_x^\varepsilon\|^2 + 2\mathbb{E}(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \mathbb{E} \|\sigma_1\|_{Q_1}^2 \\ & \leq C(1 + \mathbb{E} \|A_x^\varepsilon\|^2 + \mathbb{E} \|B_x^\varepsilon\|^2), \end{aligned}$$

it follows from Gronwall inequality that

$$\mathbb{E} \|A_x^\varepsilon(t)\|^2 \leq C(1 + \mathbb{E} \|A_{0x}\|^2 + \int_0^t \mathbb{E} \|B_x^\varepsilon(s)\|^2 ds). \quad (4.9)$$

★ For B^ε , we apply the generalized Itô formula with $\frac{1}{2} \|B_x^\varepsilon\|^2$ and obtain that

$$\begin{aligned} d\|B_x^\varepsilon\|^2 &= \frac{2}{\varepsilon} (-B_{xx}^\varepsilon, \mathcal{L}(B^\varepsilon) + \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + \eta B^\varepsilon + i\kappa A_x^\varepsilon) dt \\ &\quad + \frac{2}{\sqrt{\varepsilon}} (-B_{xx}^\varepsilon, \sigma_2 dW_2) + \frac{1}{\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 dt \\ &= -\frac{2\beta}{\varepsilon} \|B_{xx}^\varepsilon\|^2 dt + \frac{2\eta}{\varepsilon} \|B_x^\varepsilon\|^2 dt + \frac{2}{\varepsilon} (B_x^\varepsilon, i\kappa A_x^\varepsilon) dt \\ &\quad + \frac{2}{\varepsilon} (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) dt \\ &\quad + \frac{2}{\sqrt{\varepsilon}} (-B_{xx}^\varepsilon, \sigma_2 dW_2) + \frac{1}{\varepsilon} \|\sigma_2\|_{Q_2}^2 dt, \end{aligned} \quad (4.10)$$

namely, we have

$$\begin{aligned} \|B_x^\varepsilon\|^2 &= \|B_{0x}\|^2 - \frac{2\beta}{\varepsilon} \int_0^t \|B_{xx}^\varepsilon\|^2 ds + \frac{2\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \\ &\quad + \frac{2}{\varepsilon} \int_0^t (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) ds \\ &\quad + \frac{2}{\sqrt{\varepsilon}} \int_0^t (-B_{xx}^\varepsilon, \sigma_2 dW_2) + \frac{1}{\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 ds, \end{aligned} \quad (4.11)$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned} \mathbb{E} \|B_x^\varepsilon\|^2 &= \|B_{0x}\|^2 - \frac{2\beta}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds + \frac{2\eta}{\varepsilon} \int_0^t \mathbb{E} \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \mathbb{E} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \\ &\quad + \frac{2}{\varepsilon} \int_0^t \mathbb{E} (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) ds + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|_{Q_2}^2 ds, \end{aligned} \quad (4.12)$$

it is easy to see

$$\begin{aligned} & \frac{d}{dt} \mathbb{E} \|B_x^\varepsilon\|^2 \\ &= -\frac{2\beta}{\varepsilon} \mathbb{E} \|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} (B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{2}{\varepsilon} \mathbb{E} (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2. \end{aligned}$$

According to Lemma 2.6, we have

$$(-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) \leq 0,$$

it holds that

$$\begin{aligned} & \frac{d}{dt} \mathbb{E} \|B_x^\varepsilon\|^2 \\ &= -\frac{2\beta}{\varepsilon} \mathbb{E} \|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} (B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{2}{\varepsilon} \mathbb{E} (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2 \\ &\leq -\frac{2\beta}{\varepsilon} \mathbb{E} \|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} (B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2 \\ &\leq -\frac{2\beta\lambda}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{\beta\lambda}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{\kappa^2}{\varepsilon\beta\lambda} \mathbb{E} \|A_x^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2 \\ &= -\frac{\beta\lambda}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{\kappa^2}{\varepsilon\beta\lambda} \mathbb{E} \|A_x^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2 \\ &= -\frac{1}{\varepsilon} (\beta\lambda - 2\eta) \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{\kappa^2}{\varepsilon\beta\lambda} \mathbb{E} \|A_x^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|_{Q_2}^2 \\ &= -\frac{2\alpha}{\varepsilon} \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{C}{\varepsilon} (1 + \mathbb{E} \|A_x^\varepsilon\|^2), \end{aligned}$$

hence, by applying Lemma 2.3 with $\mathbb{E}\|B_x^\varepsilon\|^2$, we have

$$\begin{aligned}\mathbb{E}\|B_x^\varepsilon(t)\|^2 &\leq e^{-\frac{2\alpha}{\varepsilon}t}\mathbb{E}\|B_{0x}\|^2 + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}(1 + \mathbb{E}\|A_x^\varepsilon(s)\|^2)ds \\ &= e^{-\frac{2\alpha}{\varepsilon}t}\mathbb{E}\|B_{0x}\|^2 + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}ds + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}\mathbb{E}\|A_x^\varepsilon(s)\|^2ds \\ &\leq C(\mathbb{E}\|B_{0x}\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}\mathbb{E}\|A_x^\varepsilon(s)\|^2ds.\end{aligned}$$

Combining this and (4.9), we have

$$\begin{aligned}\mathbb{E}\|B_x^\varepsilon(t)\|^2 &\leq C(\mathbb{E}\|B_{0x}\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}\mathbb{E}\|A_x^\varepsilon\|^2ds \\ &\leq C(\mathbb{E}\|B_{0x}\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}(1 + \mathbb{E}\|A_{0x}\|^2 + \int_0^s \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau)ds \\ &\leq C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)} \int_0^s \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau ds \\ &= C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + \frac{C}{\varepsilon} \int_0^t \int_0^s e^{-\frac{2\alpha}{\varepsilon}(t-s)}\mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau ds \\ &= C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + \frac{C}{\varepsilon} \int_0^t \int_0^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}\mathbb{E}\|B_x^\varepsilon(\tau)\|^2dsd\tau \\ &= C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + \frac{C}{\varepsilon} \int_0^t \int_\tau^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}ds \cdot \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau \\ &= C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + C \int_0^t \int_0^{\frac{t-\tau}{\varepsilon}} e^{-2\alpha s}ds \cdot \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau \\ &= C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + C \int_0^t \frac{1}{2\alpha}(1 - e^{-\frac{2\alpha}{\varepsilon}(t-\tau)}) \cdot \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau \\ &\leq C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2) + C(\alpha) \int_0^t \mathbb{E}\|B_x^\varepsilon(\tau)\|^2d\tau,\end{aligned}$$

it follows from Gronwall inequality that

$$\sup_{t \in [0, \xi]} \mathbb{E}\|B_x^\varepsilon(t)\|^2 \leq C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2). \quad (4.13)$$

By replacing this estimate in (4.9), we have

$$\sup_{t \in [0, \xi]} \mathbb{E}\|A_x^\varepsilon(t)\|^2 \leq C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2). \quad (4.14)$$

- The estimates of $\mathbb{E} \sup_{t \in [0, \xi]} \|A^\varepsilon\|_{H^1}^2$ and $\mathbb{E} \sup_{t \in [0, \xi]} \|B^\varepsilon\|_{H^1}^2$.

★ It follows from (4.8) and Lemma 2.6 that

$$\begin{aligned}&\|A_x^\varepsilon\|^2 + 2\beta \int_0^t \|A_{xx}^\varepsilon\|^2ds \\ &= \|A_{0x}\|^2 + 2\eta \int_0^t \|A_x^\varepsilon\|^2ds + 2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon)ds \\ &\quad + 2 \int_0^t (-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon))ds + 2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds \\ &\leq \|A_{0x}\|^2 + 2\eta \int_0^t \|A_x^\varepsilon\|^2ds + 2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon)ds + 2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds,\end{aligned}$$

by the Cauchy inequality, we have

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \xi]} |2\eta \int_0^t \|A_x^\varepsilon\|^2 ds| \\
&= \mathbb{E} \sup_{t \in [0, \xi]} |2\eta \int_0^t (-A_{xx}^\varepsilon, A^\varepsilon) ds| \\
&\leq \frac{\eta}{2} \mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 dt + C(\eta) \mathbb{E} \int_0^\xi \|A^\varepsilon\|^2 dt, \\
& \mathbb{E} \sup_{t \in [0, \xi]} |2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon) ds| \\
&= \mathbb{E} \sup_{t \in [0, \xi]} |2 \int_0^t (-A_{xx}^\varepsilon, i\kappa B^\varepsilon) ds| \\
&\leq \frac{\eta}{2} \mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 + C(\eta) \mathbb{E} \int_0^\xi \|B^\varepsilon\|^2 dt,
\end{aligned}$$

in view of the Burkholder-Davis-Gundy inequality, it holds that

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \xi]} |2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1)| \\
&\leq \frac{\eta}{2} \mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 + C(T, \sigma_1).
\end{aligned}$$

Thus, we have

$$\mathbb{E} \sup_{t \in [0, \xi]} \|A_x^\varepsilon\|^2 \leq \|A_{0x}\|^2 + C \mathbb{E} \int_0^\xi \|A^\varepsilon\|^2 dt + C \mathbb{E} \int_0^\xi \|B^\varepsilon\|^2 dt + C,$$

according to (4.13) and (4.14), it holds that

$$\mathbb{E} \sup_{t \in [0, \xi]} \|A_x^\varepsilon\|^2 \leq C(1 + \|B_{0x}\|^2 + \|A_{0x}\|^2).$$

★ It follows from (4.11) and Lemma 2.6 that

$$\begin{aligned}
& \|B_x^\varepsilon\|^2 + \frac{2\beta}{\varepsilon} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\
&= \|B_{0x}\|^2 + \frac{2\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \\
&\quad + \frac{2}{\varepsilon} \int_0^t (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) ds \\
&\quad + \frac{2}{\sqrt{\varepsilon}} \int_0^t (-B_{xx}^\varepsilon, \sigma_2 dW_2) + \frac{1}{\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 ds \\
&\leq \|B_{0x}\|^2 + \frac{2\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \\
&\quad + \frac{2}{\sqrt{\varepsilon}} \int_0^t (-B_{xx}^\varepsilon, \sigma_2 dW_2) + \frac{1}{\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 ds,
\end{aligned}$$

by the Cauchy inequality, we have

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \xi]} \left| \frac{2\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^2 ds \right| \\
&= \mathbb{E} \sup_{t \in [0, \xi]} \left| \frac{2\eta}{\varepsilon} \int_0^t (-B_{xx}^\varepsilon, B^\varepsilon) ds \right| \\
&\leq \frac{\eta}{\varepsilon} \mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|B^\varepsilon\|^2 dt \\
& \mathbb{E} \sup_{t \in [0, \xi]} \left| \frac{2}{\varepsilon} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \right| \\
&= \mathbb{E} \sup_{t \in [0, \xi]} \left| \frac{2}{\varepsilon} \int_0^t (-B_{xx}^\varepsilon, i\kappa A^\varepsilon) ds \right| \\
&\leq \frac{\eta}{\varepsilon} \mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|A^\varepsilon\|^2 dt,
\end{aligned}$$

in view of the Burkholder-Davis-Gundy inequality, it holds that

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, \xi]} \left| \frac{2}{\sqrt{\varepsilon}} \int_0^t (-B_{xx}^\varepsilon, \sigma_2 dW_2) \right| \\
&\leq \frac{\eta}{\sqrt{\varepsilon}} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 + \frac{C(T, \sigma_2)}{\sqrt{\varepsilon}} \\
&\leq \frac{\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 + \frac{C(T, \sigma_2)}{\varepsilon},
\end{aligned}$$

in the last inequality, we have used $\varepsilon \in (0, 1)$.

Thus, we have

$$\mathbb{E} \sup_{t \in [0, \xi]} \|B_x^\varepsilon\|^2 \leq \|B_{0x}\|^2 + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|A^\varepsilon\|^2 dt + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|B^\varepsilon\|^2 dt + \frac{C}{\varepsilon},$$

according to (4.13) and (4.14), it holds that

$$\mathbb{E} \sup_{t \in [0, \xi]} \|B_x^\varepsilon\|^2 \leq \frac{C}{\varepsilon} (1 + \|B_{0x}\|^2 + \|A_{0x}\|^2).$$

- The estimates of $\mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 dt$ and $\mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt$.
- ★ Indeed, according to (4.8) and Lemma 2.6, we have

$$\begin{aligned}
& \|A_x^\varepsilon\|^2 + 2\beta \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
&= \|A_{0x}\|^2 + 2\eta \int_0^t \|A_x^\varepsilon\|^2 ds + 2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon) ds \\
&\quad + 2 \int_0^t (-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) ds + 2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds \\
&\leq \|A_{0x}\|^2 + 2\eta \int_0^t \|A_x^\varepsilon\|^2 ds + 2 \int_0^t (A_x^\varepsilon, i\kappa B_x^\varepsilon) ds + 2 \int_0^t (-A_{xx}^\varepsilon, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds,
\end{aligned}$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned} & 2\beta \mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 dt \\ & \leq \mathbb{E} \|A_{0x}\|^2 + 2\eta \mathbb{E} \int_0^\xi \|A_x^\varepsilon\|^2 dt + 2\mathbb{E} \int_0^\xi (A_x^\varepsilon, i\kappa B_x^\varepsilon) dt + \mathbb{E} \int_0^\xi \|\sigma_1\|_{Q_1}^2 dt. \end{aligned}$$

It follows from (4.13) and (4.14) that

$$\mathbb{E} \int_0^\xi \|A_{xx}^\varepsilon\|^2 dt \leq C(1 + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^2). \quad (4.15)$$

★ It follows from (4.12) and Lemma 2.6 that

$$\begin{aligned} & \mathbb{E} \|B_x^\varepsilon\|^2 + \frac{2\beta}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\ & = \|B_{0x}\|^2 + \frac{2\eta}{\varepsilon} \int_0^t \mathbb{E} \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \mathbb{E} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds \\ & \quad + \frac{2}{\varepsilon} \int_0^t \mathbb{E} (-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) ds + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|_{Q_2}^2 ds \\ & \leq \|B_{0x}\|^2 + \frac{2\eta}{\varepsilon} \int_0^t \mathbb{E} \|B_x^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \mathbb{E} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|_{Q_2}^2 ds, \end{aligned}$$

thus, we have

$$\begin{aligned} & 2\beta \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\ & \leq \varepsilon \|B_{0x}\|^2 + 2\eta \int_0^t \mathbb{E} \|B_x^\varepsilon\|^2 ds + 2\mathbb{E} \int_0^t (B_x^\varepsilon, i\kappa A_x^\varepsilon) ds + \mathbb{E} \int_0^t \|\sigma_2\|_{Q_2}^2 ds, \end{aligned}$$

namely, it holds that

$$\mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt \leq C(1 + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^2). \quad (4.16)$$

□

4.1.3 Proof of Theorem 1.1

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. For any fixed $\varepsilon \in (0, 1)$, by the Chebyshev inequality, Lemma 4.2 and the

definition of $(A^\varepsilon, B^\varepsilon)$, we have

$$\begin{aligned}
& \mathbb{P}(\{\omega \in \Omega | \tau_\infty(\omega) < +\infty\}) \\
&= \lim_{T \rightarrow +\infty} \mathbb{P}(\{\omega \in \Omega | \tau_\infty(\omega) \leq T\}) \\
&= \lim_{T \rightarrow +\infty} \mathbb{P}(\{\omega \in \Omega | \tau(\omega) = \tau_\infty(\omega)\}) \\
&= \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \mathbb{P}(\{\omega \in \Omega | \tau_R(\omega) \leq \tau(\omega)\}) \\
&= \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \mathbb{P}(\{\omega \in \Omega | \|(A^\varepsilon, B^\varepsilon)\|_{Y_\tau} \geq \|(A^\varepsilon, B^\varepsilon)\|_{Y_{\tau_R}}\}) \\
&= \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \mathbb{P}(\{\omega \in \Omega | \|(A^\varepsilon, B^\varepsilon)\|_{Y_\tau} \geq R\}) \\
&\leq \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \frac{\mathbb{E}\|(A^\varepsilon, B^\varepsilon)\|_{Y_\tau}^2}{R^2} = 0 \\
&= \lim_{T \rightarrow +\infty} \lim_{R \rightarrow +\infty} \frac{\|(A^\varepsilon, B^\varepsilon)\|_{X_{2,\tau}}^2}{R^2} = 0,
\end{aligned}$$

this shows that

$$\mathbb{P}(\{\omega \in \Omega | \tau_\infty(\omega) = +\infty\}) = 1,$$

namely, $\tau_\infty = +\infty$ P-a.s. □

4.1.4 Some a priori estimates for the slow-fast system (1.1)

Next, we establish some a priori estimates for the slow-fast system (1.1).

Proposition 4.1. *If $A_0, B_0 \in H_0^1(I)$, for $\varepsilon \in (0, 1)$, $(A^\varepsilon, B^\varepsilon)$ is the unique solution to (1.1), then for any $p > 0$, there exists a constant C_1 such that the solutions $(A^\varepsilon, B^\varepsilon)$ satisfy*

$$\begin{aligned}
& \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}\|A^\varepsilon(t)\|^{2p} \leq C, \\
& \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}\|B^\varepsilon(t)\|^{2p} \leq C, \\
& \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E}\|A^\varepsilon(t)\|_{H^1}^{2p} \leq C, \\
& \sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^T \|B^\varepsilon(t)\|_{H^1}^{2p} dt \leq C, \\
& \sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t)\|_{H^1}^{2p} \leq C,
\end{aligned}$$

where C is dependent of p, T, A_0, B_0 but independent of $\varepsilon \in (0, 1)$.

Proof. The proof of Proposition 4.1 is divided into several steps. It is also suffice to prove Proposition 4.1 holds when p is large enough. Here, the method of the proof is inspired from [12, 13, 14, 15, 16].

- The estimates of $\sup_{0 \leq t \leq T} \mathbb{E}\|A^\varepsilon(t)\|^{2p}$ and $\sup_{0 \leq t \leq T} \mathbb{E}\|B^\varepsilon(t)\|^{2p}$.

★ Indeed, we apply the generalized Itô formula (see [10, 11, 31]) with $\|A^\varepsilon\|^{2p}$ and obtain that

$$\begin{aligned}
d\|A^\varepsilon\|^{2p} &= 2p\|A^\varepsilon\|^{2p-2}(\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + \eta A^\varepsilon + i\kappa B^\varepsilon, A^\varepsilon)dt \\
&\quad + p\|A^\varepsilon\|^{2p-2}\|\sigma_1\|^2 dt + 2p(p-1)\|A^\varepsilon\|^{2p-4}(A^\varepsilon, \sigma_1 dW_1)^2 + 2p\|A^\varepsilon\|^{2p-2}(A^\varepsilon, \sigma_1 dW_1),
\end{aligned}$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned}
& \mathbb{E}\|A^\varepsilon(t)\|^{2p} \\
&= \mathbb{E}\|A^\varepsilon(0)\|^{2p} + 2p\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-2} (\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + \eta A^\varepsilon + i\kappa B^\varepsilon, A^\varepsilon) ds \\
&\quad + p\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-2} \|\sigma_1\|^2 ds + 2p(p-1)\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-4} (A^\varepsilon, \sigma_1 dW_1)^2 \\
&= \mathbb{E}\|A^\varepsilon(0)\|^{2p} + 2p\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-2} (-\beta\|A_x^\varepsilon\|^2 - \|A^\varepsilon\|_{L^4}^4 - \|A^\varepsilon\|_{L^6}^6) ds \\
&\quad + 2p\eta\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p} ds + 2p\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-2} (i\kappa B^\varepsilon, A^\varepsilon) ds \\
&\quad + p\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-2} \|\sigma_1\|^2 ds + 2p(p-1)\mathbb{E} \int_0^t \|A^\varepsilon\|^{2p-4} \|\sigma_1\sqrt{Q_1}A^\varepsilon\|^2 ds,
\end{aligned}$$

it is easy to see

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E}\|A^\varepsilon(t)\|^{2p} \\
&= 2p\mathbb{E}\|A^\varepsilon\|^{2p-2} (-\beta\|A_x^\varepsilon\|^2 - \|A^\varepsilon\|_{L^4}^4 - \|A^\varepsilon\|_{L^6}^6) \\
&\quad + 2p\eta\mathbb{E}\|A^\varepsilon\|^{2p} + 2p\mathbb{E}\|A^\varepsilon\|^{2p-2} (i\kappa B^\varepsilon, A^\varepsilon) \\
&\quad + p\mathbb{E}\|A^\varepsilon\|^{2p-2} \|\sigma_1\|^2 + 2p(p-1)\mathbb{E}\|A^\varepsilon\|^{2p-4} \|\sigma_1\sqrt{Q_1}A^\varepsilon\|^2 \\
&\leq C(p)\mathbb{E}\|B^\varepsilon\|^{2p} + C(p)\mathbb{E}\|A^\varepsilon\|^{2p} + C(p),
\end{aligned}$$

thus, we have

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E}\|A^\varepsilon(t)\|^{2p} \\
&\leq 2p\eta\mathbb{E}\|A^\varepsilon\|^{2p} + 2p\mathbb{E}\|A^\varepsilon\|^{2p-2} (i\kappa B^\varepsilon, A^\varepsilon) \\
&\quad + p\mathbb{E}\|A^\varepsilon\|^{2p-2} \|\sigma_1\|^2 + 2p(p-1)\mathbb{E}\|A^\varepsilon\|^{2p-4} \|\sigma_1\sqrt{Q_1}A^\varepsilon\|^2 \\
&\leq C(p)\mathbb{E}\|B^\varepsilon\|^{2p} + C(p)\mathbb{E}\|A^\varepsilon\|^{2p} + C(p).
\end{aligned}$$

It follows from the Gronwall inequality that

$$\mathbb{E}\|A^\varepsilon(t)\|^{2p} \leq C(1 + \|A^\varepsilon(0)\|^{2p} + \int_0^t \mathbb{E}\|B^\varepsilon(s)\|^{2p} ds).$$

★ We apply the generalized Itô formula with $\|B^\varepsilon\|^{2p}$ and obtain that

$$\begin{aligned}
d\|B^\varepsilon\|^{2p} &= \frac{2p}{\varepsilon} \|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + \eta B^\varepsilon + i\kappa A^\varepsilon, B^\varepsilon) dt \\
&\quad + \frac{p}{\varepsilon} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 dt + \frac{2p(p-1)}{\varepsilon} \|B^\varepsilon\|^{2p-4} (B^\varepsilon, \sigma_2 dW_2)^2 + \frac{2p}{\sqrt{\varepsilon}} \|B^\varepsilon\|^{2p-2} (B^\varepsilon, \sigma_2 dW_2),
\end{aligned}$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned}
& \mathbb{E}\|B^\varepsilon(t)\|^{2p} \\
&= \mathbb{E}\|B^\varepsilon(0)\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + \eta B^\varepsilon + i\kappa A^\varepsilon, B^\varepsilon) ds \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 ds + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-4} (B^\varepsilon, \sigma_2 dW_2)^2 \\
&= \mathbb{E}\|B^\varepsilon(0)\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} (-\beta\|B_x^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - \|B^\varepsilon\|_{L^6}^6) ds \\
&\quad + \frac{2p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p} ds + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} (i\kappa A^\varepsilon, B^\varepsilon) ds \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 ds + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-4} \|\sigma_2\sqrt{Q_2}B^\varepsilon\|^2 ds,
\end{aligned}$$

it is easy to see

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^{2p} \\
&= \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (-\beta \|B_x^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - \|B^\varepsilon\|_{L^6}^6) \\
&\quad + \frac{2p\eta}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (i\kappa A^\varepsilon, B^\varepsilon) \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-4} \|\sigma_2 \sqrt{Q_2} B^\varepsilon\|^2,
\end{aligned}$$

thus, we have

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^{2p} \\
&\leq \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (-\beta \|B_x^\varepsilon\|^2) \\
&\quad + \frac{2p\eta}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (i\kappa A^\varepsilon, B^\varepsilon) \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-4} \|\sigma_2 \sqrt{Q_2} B^\varepsilon\|^2 \\
&\leq -\frac{2p\beta\lambda}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} \\
&\quad + \frac{2p\eta}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (i\kappa A^\varepsilon, B^\varepsilon) \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-4} \|\sigma_2 \sqrt{Q_2} B^\varepsilon\|^2 \\
&= \frac{-2p(\beta\lambda-\eta)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} (i\kappa A^\varepsilon, B^\varepsilon) \\
&\quad + \frac{p}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p-4} \|\sigma_2 \sqrt{Q_2} B^\varepsilon\|^2.
\end{aligned}$$

By using the Young inequality (see Lemma 2.2), we have

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^{2p} \\
&\leq \frac{-2p(\frac{\beta\lambda}{2}-\eta)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon} \\
&\leq \frac{-2p\alpha}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon}.
\end{aligned}$$

Hence, by applying Lemma 2.3 with $\mathbb{E} \|B^\varepsilon(t)\|^{2p}$, we have

$$\begin{aligned}
& \mathbb{E} \|B^\varepsilon(t)\|^{2p} \\
&\leq \mathbb{E} \|B^\varepsilon(0)\|^{2p} e^{-\frac{2p\alpha}{\varepsilon}t} + \frac{C(p)}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} (\mathbb{E} \|A^\varepsilon(s)\|^{2p} + 1) ds.
\end{aligned}$$

Thus, by the same method in the above, we have

$$\begin{aligned}
& \mathbb{E}\|B^\varepsilon(t)\|^{2p} \\
& \leq \|B_0\|^{2p} e^{-\frac{2p\alpha}{\varepsilon}t} + \frac{C(p)}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} ds + \frac{C(p)}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|^{2p} ds \\
& \leq C(1 + \|B_0\|^{2p}) + \frac{C(p)}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|^{2p} ds \\
& \leq C(1 + \|B_0\|^{2p}) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} [1 + \|A^\varepsilon(0)\|^{2p} + \int_0^s \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau] ds \\
& = C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} \int_0^s \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau ds \\
& \leq C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + \frac{C}{\varepsilon} \int_0^t \int_0^s e^{-\frac{2p\alpha}{\varepsilon}(t-s)} \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau ds \\
& = C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + \frac{C}{\varepsilon} \int_0^t \int_\tau^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} ds d\tau \\
& = C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + \frac{C}{\varepsilon} \int_0^t \int_\tau^t e^{-\frac{2p\alpha}{\varepsilon}(t-s)} ds \cdot \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau \\
& = C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + C \int_0^t \int_0^{\frac{t-\tau}{\varepsilon}} e^{-2p\alpha s} ds \cdot \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau \\
& = C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + C \int_0^t \frac{1}{2p\alpha} (1 - e^{-\frac{2p\alpha}{\varepsilon}(t-\tau)}) \cdot \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau \\
& \leq C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}) + C \int_0^t \mathbb{E}\|B^\varepsilon(\tau)\|^{2p} d\tau,
\end{aligned}$$

thus, it follows from Gronwall inequality that

$$\mathbb{E}\|B^\varepsilon(t)\|^{2p} \leq C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}).$$

Moreover, we have

$$\mathbb{E}\|A^\varepsilon(t)\|^{2p} \leq C(1 + \|A_0\|^{2p} + \|B_0\|^{2p}).$$

- The estimates of $\sup_{0 \leq t \leq T} \mathbb{E}\|A^\varepsilon(t)\|_{H^1}^{2p}$ and $\mathbb{E} \int_0^T \|B_x^\varepsilon\|^{2p} dt$.

★ Indeed, it follows from (4.7) that

$$\begin{aligned}
& d\|A_x^\varepsilon\|^2 \\
& = (-2\beta\|A_{xx}^\varepsilon\|^2 + 2\eta\|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + 2(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) + \|\sigma_1\|_{Q_1}^2) dt \\
& \quad + 2(-A_{xx}^\varepsilon, \sigma_1 dW_1),
\end{aligned}$$

then, we apply the generalized Itô formula (see [10, 11, 31]) with $\|A_x^\varepsilon\|^{2p}$ and obtain that

$$\begin{aligned}
& d\|A_x^\varepsilon\|^{2p} \\
& = p\|A_x^\varepsilon\|^{2p-2} \{ -2\beta\|A_{xx}^\varepsilon\|^2 + 2\eta\|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + 2(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) + \|\sigma_1\|_{Q_1}^2 \} dt \\
& \quad + 2p(p-1)\|A_x^\varepsilon\|^{2p-4} (-A_{xx}^\varepsilon, \sigma_1 dW_1)^2 + 2p\|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1),
\end{aligned}$$

thus, we have

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
&= \|A_x^\varepsilon(0)\|^{2p} \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 + 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + 2(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) + \|\sigma_1\|_{Q_1}^2 \} ds \\
&+ 2p(p-1) \int_0^t \|A_x^\varepsilon\|^{2p-4} (-A_{xx}^\varepsilon, \sigma_1 dW_1)^2 + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1).
\end{aligned}$$

According to Lemma 2.6, we have

$$(-A_{xx}^\varepsilon, \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon)) \leq 0,$$

thus, it holds that

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
&\leq \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 + 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds \\
&+ 2p(p-1) \int_0^t \|A_x^\varepsilon\|^{2p-4} \|\sigma_1 \sqrt{Q_1} A_{xx}^\varepsilon\|^2 ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1) \\
&= \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 \} ds + 2p(p-1) \int_0^t \|A_x^\varepsilon\|^{2p-4} \|\sigma_1 \sqrt{Q_1} A_{xx}^\varepsilon\|^2 ds \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1) \\
&\leq \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 \} ds + C(\sigma_1, Q_1) p \int_0^t \|A_x^\varepsilon\|^{2p-4} \cdot (p-1) \|A_{xx}^\varepsilon\|^2 ds \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1).
\end{aligned}$$

It follows from the Young inequality (see Lemma 2.2) that

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
&\leq \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 \} ds + C(\sigma_1, Q_1) p \int_0^t [\rho \|A_x^\varepsilon\|^{2p-2} + C(p, \rho)] \|A_{xx}^\varepsilon\|^2 ds \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1) \\
&= \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ -2\beta \|A_{xx}^\varepsilon\|^2 \} ds \\
&+ C(\sigma_1, Q_1) p \rho \int_0^t \|A_x^\varepsilon\|^{2p-2} \|A_{xx}^\varepsilon\|^2 ds + C(\sigma_1, Q_1) C(p, \rho) p \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1) \\
&= \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \|A_{xx}^\varepsilon\|^2 \{ -2\beta + \rho C(\sigma_1, Q_1) \} ds + C(p, \rho, \sigma_1, Q_1) \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
&+ p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1),
\end{aligned} \tag{4.17}$$

by taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned} & \mathbb{E}\|A_x^\varepsilon(t)\|^{2p} \\ & \leq \mathbb{E}\|A_x^\varepsilon(0)\|^{2p} + p\mathbb{E}\int_0^t \|A_x^\varepsilon\|^{2p-2}\|A_{xx}^\varepsilon\|^2\{-2\beta + \rho C(\sigma_1, Q_1)\}ds + C(p, \rho, \sigma_1, Q_1)\mathbb{E}\int_0^t \|A_{xx}^\varepsilon\|^2ds \\ & \quad + p\mathbb{E}\int_0^t \|A_x^\varepsilon\|^{2p-2}\{2\eta\|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2\}ds. \end{aligned}$$

If we take $0 < \rho < 1$, we have

$$-2\beta + \rho C(\sigma_1, Q_1) < 0,$$

thus, we have

$$\begin{aligned} & \mathbb{E}\|A_x^\varepsilon(t)\|^{2p} \\ & \leq \mathbb{E}\|A_x^\varepsilon(0)\|^{2p} + C(p, \rho, \sigma_1, Q_1)\mathbb{E}\int_0^t \|A_{xx}^\varepsilon\|^2ds + p\mathbb{E}\int_0^t \|A_x^\varepsilon\|^{2p-2}\{2\eta\|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2\}ds. \end{aligned}$$

It follows from the Young inequality (see Lemma 2.2) and (4.15) that

$$\begin{aligned} & \mathbb{E}\|A_x^\varepsilon(t)\|^{2p} \\ & \leq \mathbb{E}\|A_x^\varepsilon(0)\|^{2p} + C\mathbb{E}\int_0^t \|A_{xx}^\varepsilon\|^2ds + C\mathbb{E}\int_0^t \|A_x^\varepsilon\|^{2p}ds + C\mathbb{E}\int_0^t \|B_x^\varepsilon\|^{2p}ds + C \\ & \leq C(1 + \|A_{0x}\|^{2p} + \|B_{0x}\|^{2p} + \mathbb{E}\int_0^t \|A_x^\varepsilon\|^{2p}ds + \mathbb{E}\int_0^t \|B_x^\varepsilon\|^{2p}ds), \end{aligned}$$

hence, by Gronwall inequality, we have

$$\mathbb{E}\|A_x^\varepsilon(t)\|^{2p} \leq C(1 + \|A_{0x}\|^{2p} + \|B_{0x}\|^{2p} + \mathbb{E}\int_0^t \|B_x^\varepsilon(s)\|^{2p}ds). \quad (4.18)$$

★ Indeed, it follows from (4.10) that

$$\begin{aligned} d\|B_x^\varepsilon\|^2 & = (-\frac{2\beta}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon}\|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon}(B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|_{Q_2}^2)dt \\ & \quad + \frac{2}{\sqrt{\varepsilon}}(-B_{xx}^\varepsilon, \sigma_2 dW_2), \end{aligned}$$

then, we apply the generalized Itô formula (see [10, 11, 31]) with $\|B_x^\varepsilon\|^{2p}$ and obtain that

$$\begin{aligned} & d\|B_x^\varepsilon\|^{2p} \\ & = p\|B_x^\varepsilon\|^{2p-2}\left\{-\frac{2\beta}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon}\|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon}(B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|_{Q_2}^2\right\}dt \\ & \quad + \frac{2p(p-1)}{\varepsilon}\|B_x^\varepsilon\|^{2p-4}(-B_{xx}^\varepsilon, \sigma_2 dW_2)^2 + \frac{2p}{\sqrt{\varepsilon}}\|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2), \end{aligned}$$

thus, we have

$$\begin{aligned} & \|B_x^\varepsilon(t)\|^{2p} \\ & = \|B_x^\varepsilon(0)\|^{2p} \\ & \quad + p\int_0^t \|B_x^\varepsilon\|^{2p-2}\left\{-\frac{2\beta}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2\eta}{\varepsilon}\|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon}(B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|_{Q_2}^2\right\}ds \\ & \quad + \frac{2p(p-1)}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-4}(-B_{xx}^\varepsilon, \sigma_2 dW_2)^2 + \frac{2p}{\sqrt{\varepsilon}}\int_0^t \|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2). \end{aligned}$$

According to Lemma 2.6, we have

$$(-B_{xx}^\varepsilon, \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon)) \leq 0,$$

thus, it holds that

$$\begin{aligned} & \|B_x^\varepsilon(t)\|^{2p} \\ & \leq \|B_x^\varepsilon(0)\|^{2p} + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ -\frac{2\beta}{\varepsilon} \|B_{xx}^\varepsilon\|^2 \right\} ds + \frac{2p(p-1)}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p-4} \|\sigma_2 \sqrt{Q_2} B_{xx}^\varepsilon\|^2 ds \\ & \quad + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ \frac{2\eta}{\varepsilon} \|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon} (B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{1}{\varepsilon} \|\sigma_2\|_{Q_2}^2 \right\} ds \\ & \quad + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_x^\varepsilon\|^{2p-2} (-B_{xx}^\varepsilon, \sigma_2 dW_2), \end{aligned}$$

thus, we have

$$\begin{aligned} & \|B_x^\varepsilon(t)\|^{2p} \\ & \leq \|B_x^\varepsilon(0)\|^{2p} + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ -\frac{2\beta}{\varepsilon} \|B_{xx}^\varepsilon\|^2 \right\} ds + \frac{2p(p-1)C(\sigma_2, Q_2)}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p-4} \|B_{xx}^\varepsilon\|^2 ds \\ & \quad + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ \frac{2\eta}{\varepsilon} \|B_x^\varepsilon\|^2 + \frac{2}{\varepsilon} (B_x^\varepsilon, i\kappa A_x^\varepsilon) + \frac{1}{\varepsilon} \|\sigma_2\|_{Q_2}^2 \right\} ds + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_x^\varepsilon\|^{2p-2} (-B_{xx}^\varepsilon, \sigma_2 dW_2), \end{aligned}$$

it follows from the Young inequality (see Lemma 2.2) that

$$\begin{aligned} & \|B_x^\varepsilon(t)\|^{2p} \\ & \leq \|B_x^\varepsilon(0)\|^{2p} \\ & \quad + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ -\frac{2\beta}{\varepsilon} \|B_{xx}^\varepsilon\|^2 \right\} ds + \frac{pC(\sigma_2, Q_2)}{\varepsilon} \int_0^t [\rho \|B_x^\varepsilon\|^{2p-2} + C(p, \rho)] \|B_{xx}^\varepsilon\|^2 ds \\ & \quad + \frac{3p\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C(p, T, \sigma_2, Q_2)}{\varepsilon} \left(\int_0^t \|A_x^\varepsilon\|^{2p} ds + 1 \right) + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_x^\varepsilon\|^{2p-2} (-B_{xx}^\varepsilon, \sigma_2 dW_2) \\ & = \|B_x^\varepsilon(0)\|^{2p} + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \left\{ -\frac{2\beta}{\varepsilon} \|B_{xx}^\varepsilon\|^2 \right\} ds \\ & \quad + \frac{pC(\sigma_2, Q_2)\rho}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p-2} \|B_{xx}^\varepsilon\|^2 ds + \frac{pC(p, \rho)C(\sigma_2, Q_2)}{\varepsilon} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\ & \quad + \frac{3p\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C(p, T, \sigma_2, Q_2)}{\varepsilon} \left(\int_0^t \|A_x^\varepsilon\|^{2p} ds + 1 \right) + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_x^\varepsilon\|^{2p-2} (-B_{xx}^\varepsilon, \sigma_2 dW_2) \\ & = \|B_x^\varepsilon(0)\|^{2p} \\ & \quad + p \int_0^t \|B_x^\varepsilon\|^{2p-2} \|B_{xx}^\varepsilon\|^2 \left\{ -\frac{2\beta}{\varepsilon} + \frac{\rho C(\sigma_2, Q_2)}{\varepsilon} \right\} ds + \frac{C(p, \rho, \sigma_2, Q_2)}{\varepsilon} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\ & \quad + \frac{3p\eta}{\varepsilon} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C(p, T, \sigma_2, Q_2)}{\varepsilon} \left(\int_0^t \|A_x^\varepsilon\|^{2p} ds + 1 \right) + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_x^\varepsilon\|^{2p-2} (-B_{xx}^\varepsilon, \sigma_2 dW_2). \end{aligned}$$

By taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned} & \mathbb{E} \|B_x^\varepsilon(t)\|^{2p} \leq \mathbb{E} \|B_x^\varepsilon(0)\|^{2p} \\ & \quad + p \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p-2} \left(-\frac{2\beta}{\varepsilon} + \frac{\rho C(\sigma_2, Q_2)}{\varepsilon} \right) \|B_{xx}^\varepsilon\|^2 ds + \frac{C(p, \rho, \sigma_2, Q_2)}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \\ & \quad + \frac{3p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C(p, T, \sigma_2, Q_2)}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + 1). \end{aligned}$$

It follows from (4.16) that

$$\begin{aligned}
& \mathbb{E} \|B_x^\varepsilon(t)\|^{2p} \\
& \leq \|B_0\|_{H^1}^{2p} + p \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p-2} \left(-\frac{2\beta}{\varepsilon} + \frac{\rho C(\sigma_2, Q_2)}{\varepsilon} \right) \|B_{xx}^\varepsilon\|^2 ds + \frac{3p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds \\
& \quad + \frac{C(p, T, \sigma_2, Q_2)}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^2 + 1).
\end{aligned}$$

If we take $0 < \rho < 1$, we have

$$-2\beta + \rho C(\sigma_2, Q_2) < -\frac{3\beta}{2},$$

thus, it holds that

$$\begin{aligned}
& \mathbb{E} \|B_x^\varepsilon(t)\|^{2p} \\
& \leq \|B_0\|_{H^1}^{2p} - \frac{3\beta p}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p-2} \|B_{xx}^\varepsilon\|^2 ds + \frac{3p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds \\
& \quad + \frac{C}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^2 + 1) \\
& \leq -\frac{3\beta \lambda p}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{3p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds \\
& \quad + \frac{C}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^{2p} + 1) \\
& \leq \frac{-\frac{3}{2}p\beta\lambda + 3p\eta}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^{2p} + 1) \\
& \leq \frac{-3p\alpha}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + \|A_0\|_{H^1}^2 + \|B_0\|_{H^1}^{2p} + 1) \\
& \leq \frac{-3p\alpha}{\varepsilon} \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds + \frac{C}{\varepsilon} (\mathbb{E} \int_0^t \|A_x^\varepsilon\|^{2p} ds + 1).
\end{aligned}$$

Hence, by applying Lemma 2.3 with $\mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds$, we have

$$\begin{aligned}
& \mathbb{E} \int_0^t \|B_x^\varepsilon\|^{2p} ds \\
& \leq \int_0^t e^{-\frac{3p\alpha}{\varepsilon}(t-s)} \frac{C}{\varepsilon} \left(\int_0^s \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + 1 \right) ds \\
& = \frac{C}{\varepsilon} \int_0^t e^{-\frac{3p\alpha}{\varepsilon}(t-s)} \int_0^s \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau ds + \frac{C}{\varepsilon} \int_0^t e^{-\frac{3p\alpha}{\varepsilon}(t-s)} ds \\
& \leq \frac{C}{\varepsilon} \int_0^t \int_0^s e^{-\frac{3p\alpha}{\varepsilon}(t-s)} \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau ds + C \\
& = \frac{C}{\varepsilon} \int_0^t \int_\tau^t e^{-\frac{3p\alpha}{\varepsilon}(t-s)} \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} ds d\tau + C \\
& = \frac{C}{\varepsilon} \int_0^t \int_\tau^t e^{-\frac{3p\alpha}{\varepsilon}(t-s)} ds \cdot \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + C \\
& = C \int_0^t \int_0^{\frac{t-\tau}{\varepsilon}} e^{-3p\alpha s} ds \cdot \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + C \\
& = C \int_0^t \frac{1}{3p\alpha} (1 - e^{-\frac{3p\alpha}{\varepsilon}(t-\tau)}) \cdot \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + C \\
& \leq C \int_0^t \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + C,
\end{aligned}$$

plug this inequality into (4.18), we have

$$\mathbb{E} \|A_x^\varepsilon(t)\|^{2p} \leq C(1 + \|A_{0x}\|^{2p} + \|B_{0x}\|^{2p} + C \int_0^t \mathbb{E} \|A_x^\varepsilon(\tau)\|^{2p} d\tau + C),$$

by using Gronwall inequality, we have

$$\sup_{0 \leq t \leq T} \mathbb{E} \|A_x^\varepsilon(t)\|^{2p} \leq C(1 + \|B_{0x}\|^{2p} + \|A_{0x}\|^{2p}), \quad (4.19)$$

moreover, we have

$$\mathbb{E} \int_0^T \|B_x^\varepsilon\|^{2p} dt \leq C(1 + \|B_{0x}\|^{2p} + \|A_{0x}\|^{2p}). \quad (4.20)$$

- The estimate of $\mathbb{E} \sup_{0 \leq t \leq T} \|A_x^\varepsilon(t)\|^{2p}$.

Indeed, it follows from (4.17) that

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
& \leq \|A_x^\varepsilon(0)\|^{2p} + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \|A_{xx}^\varepsilon\|^2 \{ -2\beta + \rho C(\sigma_1, Q_1) \} ds + C(p, \rho, \sigma_1, Q_1) \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
& \quad + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{ 2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2 \} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1).
\end{aligned}$$

If we take $0 < \rho < 1$, we have

$$-2\beta + \rho C(\sigma_1, Q_1) < 0,$$

thus, we have

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
& \leq \|A_x^\varepsilon(0)\|^{2p} + C(p, \rho, \sigma_1, Q_1) \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
& \quad + p \int_0^t \|A_x^\varepsilon\|^{2p-2} \{2\eta \|A_x^\varepsilon\|^2 + 2(A_x^\varepsilon, i\kappa B_x^\varepsilon) + \|\sigma_1\|_{Q_1}^2\} ds + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1).
\end{aligned}$$

It follows from the Young inequality (see Lemma 2.2) that

$$\begin{aligned}
& \|A_x^\varepsilon(t)\|^{2p} \\
& \leq \|A_x^\varepsilon(0)\|^{2p} + C \int_0^t \|A_{xx}^\varepsilon\|^2 ds \\
& \quad + C \int_0^t \|A_x^\varepsilon\|^{2p} ds + C \int_0^t \|B_x^\varepsilon\|^{2p} ds + C + 2p \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1).
\end{aligned}$$

In view of the Burkholder-Davis-Gundy inequality as in [42, P87], it holds that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \|A_x^\varepsilon\|^{2p-2} (-A_{xx}^\varepsilon, \sigma_1 dW_1) \right| \\
& = \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (-A_{xx}^\varepsilon, \|A_x^\varepsilon\|^{2p-2} \sigma_1 dW_1) \right| \\
& \leq \mathbb{E} \left(\int_0^T \|A_x^\varepsilon\|^2 \|A_x^\varepsilon\|^{4p-4} dt \right)^{\frac{1}{2}} \\
& \leq \mathbb{E} \left(\int_0^T \|A_x^\varepsilon\|^{4p-2} dt \right)^{\frac{1}{2}} \\
& \leq \mathbb{E} \int_0^T \|A_x^\varepsilon\|^{4p-2} dt + C \\
& \leq C,
\end{aligned}$$

where we have use (4.19) in the last inequality.

According to (4.15) and (4.20), it holds that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \|A_{xx}^\varepsilon\|^2 ds \right| \leq \mathbb{E} \int_0^T \|A_{xx}^\varepsilon\|^2 dt \leq C, \\
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \|B_x^\varepsilon\|^{2p} ds \right| \leq \mathbb{E} \int_0^T \|B_x^\varepsilon\|^{2p} dt \leq C.
\end{aligned}$$

It follows from the above estimates, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} \|A_x^\varepsilon(s)\|^{2p} \leq C + C \mathbb{E} \int_0^t \|A_x^\varepsilon(s)\|^{2p} ds \\
& \leq C + C \int_0^t \mathbb{E} \sup_{0 \leq \tau \leq s} \|A_x^\varepsilon(\tau)\|^{2p} ds.
\end{aligned}$$

By using Gronwall inequality, we have

$$\mathbb{E} \sup_{0 \leq s \leq T} \|A_x^\varepsilon(s)\|^{2p} \leq C.$$

□

4.2 Well-posedness for the averaged equation (1.2)

By the same method in Theorem 1.1 and Proposition 4.1, we can obtain the following proposition.

Proposition 4.2. *If $A_0 \in H_0^1(I)$, (1.2) has a unique solution $\bar{A} \in L^2(\Omega, C([0, T]; H_0^1(I)))$. Moreover, for any $p > 0$, there exists a constant C such that the solution \bar{A} satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \|\bar{A}(t)\|_{H^1}^{2p} \leq C,$$

where C dependent of p, T, A_0, B_0 but independent of $p > 0$.

5 Proof of Theorem 1.2

5.1 Hölder continuity of time variable for A^ε

The following proposition is a crucial step.

Proposition 5.1. *There exists a constant $C(p, T)$ such that*

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E} \|A^\varepsilon(t+h) - A^\varepsilon(t)\|^{2p} \leq C(p, T)h^p \quad (5.1)$$

for any $t \in [0, T], h > 0$.

Proof. Let us write

$$\begin{aligned} & A^\varepsilon(t+h) - A^\varepsilon(t) \\ &= (S(h) - Id)A^\varepsilon(t) + \int_t^{t+h} S(t+h-s)(\mathcal{F}(A^\varepsilon(s)) + \mathcal{G}(A^\varepsilon(s)) + \eta A^\varepsilon(s) + i\kappa B^\varepsilon(s))ds \\ & \quad + \int_t^{t+h} S(t+h-s)\sigma_1 dW_1(s), \end{aligned}$$

here Id denotes the identity operator.

Due to [30], there is a C such that for all $x \in H^1(I)$,

$$\|(S(h) - Id)x\| \leq Ch^{\frac{1}{2}}\|x\|_{H^1},$$

and then, according to this inequality and Proposition 4.1, we have

$$\mathbb{E} \|(S(h) - Id)A^\varepsilon(t)\|^{2p} \leq Ch^p \mathbb{E} \|A^\varepsilon(t)\|_{H^1}^{2p} \leq Ch^p.$$

It follows from Proposition 4.1 that

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} S(t+h-s)\mathcal{F}(A^\varepsilon(s))ds \right\|^{2p} \\ & \leq \mathbb{E} \left(\int_t^{t+h} \|S(t+h-s)\mathcal{F}(A^\varepsilon(s))\|ds \right)^{2p} \\ & \leq \mathbb{E} \left(\int_t^{t+h} \|\mathcal{F}(A^\varepsilon(s))\|ds \right)^{2p} \\ & \leq \mathbb{E} \left[\left(\int_t^{t+h} 1ds \right)^{2p-1} \cdot \int_t^{t+h} \|\mathcal{F}(A^\varepsilon(s))\|^{2p}ds \right] \\ & \leq h^{2p-1} \mathbb{E} \left(\int_t^{t+h} \|\mathcal{F}(A^\varepsilon(s))\|^{2p}ds \right) \\ & = h^{2p-1} \int_t^{t+h} \mathbb{E} \|\mathcal{F}(A^\varepsilon(s))\|^{2p}ds, \end{aligned}$$

according to the facts

$$\|\mathcal{F}(A^\varepsilon)\| = C\|A^\varepsilon\|_{L^6}^3 \leq C\|A^\varepsilon\|_{H^1}^3,$$

it holds that

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} S(t+h-s) \mathcal{F}(A^\varepsilon(s)) ds \right\|^{2p} \\ & \leq Ch^{2p-1} \int_t^{t+h} \mathbb{E} \|A^\varepsilon(s)\|_{H^1}^{6p} ds \\ & \leq Ch^{2p}, \end{aligned}$$

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} S(t+h-s) \mathcal{G}(A^\varepsilon(s)) ds \right\|^{2p} \\ & \leq \mathbb{E} \left(\int_t^{t+h} \|S(t+h-s) \mathcal{G}(A^\varepsilon(s))\| ds \right)^{2p} \\ & \leq \mathbb{E} \left(\int_t^{t+h} \|\mathcal{G}(A^\varepsilon(s))\| ds \right)^{2p} \\ & \leq \mathbb{E} \left[\left(\int_t^{t+h} 1 ds \right)^{2p-1} \cdot \int_t^{t+h} \|\mathcal{G}(A^\varepsilon(s))\|^{2p} ds \right] \\ & \leq h^{2p-1} \mathbb{E} \left(\int_t^{t+h} \|\mathcal{G}(A^\varepsilon(s))\|^{2p} ds \right) \\ & = h^{2p-1} \int_t^{t+h} \mathbb{E} \|\mathcal{G}(A^\varepsilon(s))\|^{2p} ds \end{aligned}$$

according to the facts

$$\|\mathcal{G}(A^\varepsilon)\| = \|A^\varepsilon\|_{L^{10}}^5 \leq C\|A^\varepsilon\|_{H^1}^5,$$

it holds that

$$\begin{aligned} & \mathbb{E} \left\| \int_t^{t+h} S(t+h-s) \mathcal{G}(A^\varepsilon(s)) ds \right\|^{2p} \\ & \leq Ch^{2p-1} \int_t^{t+h} \mathbb{E} \|A^\varepsilon(s)\|_{H^1}^{10p} ds \\ & \leq Ch^{2p}. \end{aligned}$$

By the same way, we have

$$\begin{aligned}
& \mathbb{E} \left\| \int_t^{t+h} S(t+h-s)(\eta A^\varepsilon(s)) ds \right\|^{2p} \\
& \leq \mathbb{E} \left(\int_t^{t+h} \|S(t+h-s)(\eta A^\varepsilon(s))\| ds \right)^{2p} \\
& \leq \mathbb{E} \left(\int_t^{t+h} \|(\eta A^\varepsilon(s))\| ds \right)^{2p} \\
& \leq \mathbb{E} \left[\left(\int_t^{t+h} 1 ds \right)^{2p-1} \cdot \int_t^{t+h} \|(\eta A^\varepsilon(s))\|^{2p} ds \right] \\
& \leq h^{2p-1} \mathbb{E} \left(\int_t^{t+h} \|(\eta A^\varepsilon(s))\|^{2p} ds \right) \\
& = h^{2p-1} \int_t^{t+h} \mathbb{E} \|(\eta A^\varepsilon(s))\|^{2p} ds \\
& \leq Ch^{2p-1} \int_t^{t+h} \mathbb{E} \|A^\varepsilon(s)\|^{2p} ds \\
& \leq Ch^{2p}
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left\| \int_t^{t+h} S(t+h-s)(i\kappa B^\varepsilon(s)) ds \right\|^{2p} \\
& \leq \mathbb{E} \left(\int_t^{t+h} \|S(t+h-s)(i\kappa B^\varepsilon(s))\| ds \right)^{2p} \\
& \leq \mathbb{E} \left(\int_t^{t+h} \|(i\kappa B^\varepsilon(s))\| ds \right)^{2p} \\
& \leq \mathbb{E} \left[\left(\int_t^{t+h} 1 ds \right)^{2p-1} \cdot \int_t^{t+h} \|(i\kappa B^\varepsilon(s))\|^{2p} ds \right] \\
& \leq h^{2p-1} \mathbb{E} \left(\int_t^{t+h} \|(i\kappa B^\varepsilon(s))\|^{2p} ds \right) \\
& = h^{2p-1} \int_t^{t+h} \mathbb{E} \|(i\kappa B^\varepsilon(s))\|^{2p} ds \\
& \leq Ch^{2p-1} \int_t^{t+h} \mathbb{E} \|B^\varepsilon(s)\|^{2p} ds \\
& \leq Ch^{2p}.
\end{aligned}$$

In view of the Burkholder-Davis-Gundy inequality and Hölders inequality, it yields

$$\begin{aligned}
& \mathbb{E} \left\| \int_t^{t+h} S(t+h-s)\sigma_1 dW_1(s) \right\|^{2p} \\
& \leq C \mathbb{E} \left(\int_t^{t+h} \|S(t+h-s)\sigma_1\|_{Q_1}^2 ds \right)^p \\
& \leq C \mathbb{E} \left(\int_t^{t+h} \|\sigma_1\|_{Q_1}^2 ds \right)^p \\
& \leq Ch^p.
\end{aligned}$$

By the above estimates, we arrive at (5.1). □

5.2 Auxiliary process $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$

Next, we introduce an auxiliary process $(\hat{A}^\varepsilon, \hat{B}^\varepsilon) \in H \times H$ by Khasminskii in [29].

Fix a positive number δ and do a partition of time interval $[0, T]$ of size δ . We construct a process $\hat{B}^\varepsilon \in L^2(I)$ by means of the equations

$$\begin{aligned}\hat{B}^\varepsilon(t) &= B^\varepsilon(k\delta) + \frac{1}{\varepsilon} \int_{k\delta}^t \mathcal{L}\hat{B}^\varepsilon(s) ds \\ &\quad + \frac{1}{\varepsilon} \int_{k\delta}^t [\mathcal{F}(\hat{B}^\varepsilon(s)) + \mathcal{G}(\hat{B}^\varepsilon(s)) + \eta\hat{B}^\varepsilon(s) + i\kappa A^\varepsilon(k\delta)] ds \\ &\quad + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t \sigma_2 dW_2(s)\end{aligned}$$

for $t \in [k\delta, \min\{(k+1)\delta, T\})$, $k \geq 0$.

Also define the process $\hat{A}^\varepsilon \in L^2(I)$ by

$$\begin{aligned}\hat{A}^\varepsilon(t) &= A_0 + \int_0^t \mathcal{L}\hat{A}^\varepsilon(s) ds \\ &\quad + \int_0^t [\mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon(s(\delta))) + \eta A^\varepsilon(s(\delta)) + i\kappa \hat{B}^\varepsilon(s)] ds \\ &\quad + \int_0^t \sigma_1 dW_1(s)\end{aligned}$$

for $t \in [0, T]$, where $s(\delta) = [\frac{s}{\delta}]\delta$ is the nearest breakpoint preceding s and $[\cdot]$ is the integer function.

Thus $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$ satisfies

$$\begin{cases} d\hat{A}^\varepsilon = [\mathcal{L}(\hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon(t(\delta))) + \mathcal{G}(A^\varepsilon(t(\delta))) + \eta A^\varepsilon(t(\delta)) + i\kappa \hat{B}^\varepsilon] ds + \sigma_1 dW_1 & \text{in } Q \\ d\hat{B}^\varepsilon = \frac{1}{\varepsilon} [\mathcal{L}(\hat{B}^\varepsilon) + \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(\hat{B}^\varepsilon) + \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon(k\delta)] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2 dW_2 & \text{in } I \times (k\delta, \min\{(k+1)\delta, T\}) \\ \hat{A}^\varepsilon(0, t) = 0 = \hat{A}^\varepsilon(1, t) & \text{in } (0, T) \\ \hat{B}^\varepsilon(0, t) = 0 = \hat{B}^\varepsilon(1, t) & \text{in } (k\delta, \min\{(k+1)\delta, T\}) \\ \hat{A}^\varepsilon(x, 0) = A_0(x) & \text{in } I \\ \hat{B}^\varepsilon(x, k\delta) = B^\varepsilon(x, k\delta) & \text{in } I. \end{cases} \quad (5.2)$$

5.3 Some priori estimates of $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$

Because the proof almost follows the steps in Proposition 4.1, we omit the proof here.

Proposition 5.2. *If $A_0, B_0 \in H_0^1(I)$, for $\varepsilon \in (0, 1)$, $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$ is the unique solution to (5.2), then there exists a constant C such that the solutions $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$ satisfy*

$$\begin{aligned}\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{t \in [0, T]} \|\hat{A}^\varepsilon(t)\|_{H^1}^2 &\leq C_1, \\ \sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{t \in [0, T]} \|\hat{B}^\varepsilon(t)\|_{H^1}^2 &\leq C_1, \\ \sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^T \|\hat{A}_{xx}^\varepsilon\|^2 dt &\leq C_1, \\ \sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^T \|\hat{B}_{xx}^\varepsilon\|^2 dt &\leq C_1,\end{aligned}$$

where C_1 dependent of T, A_0, B_0 but independent of $\varepsilon \in (0, 1)$.

Moreover, for any $p > 0$, there exists a constant C_2 such that

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|\hat{A}^\varepsilon(t)\|^{2p} &\leq C_2, \\ \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|\hat{B}^\varepsilon(t)\|^{2p} &\leq C_2, \\ \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,T]} \mathbb{E} \|\hat{A}^\varepsilon(t)\|_{H^1}^{2p} &\leq C_2, \\ \sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^T \|\hat{B}^\varepsilon(t)\|_{H^1}^{2p} dt &\leq C_2, \\ \sup_{\varepsilon \in (0,1)} \mathbb{E} \sup_{t \in [0,T]} \|\hat{A}^\varepsilon(t)\|_{H^1}^{2p} &\leq C_2, \end{aligned}$$

where C_2 dependent of p, T, A_0, B_0 but independent of $\varepsilon \in (0, 1)$, $p > 0$.

5.4 The errors of $A^\varepsilon - \hat{A}^\varepsilon$ and $B^\varepsilon - \hat{B}^\varepsilon$

We will establish the convergence of the auxiliary process \hat{B}^ε to the fast solution process B^ε and \hat{A}^ε to the slow solution process A^ε , respectively.

Lemma 5.1. *There exists a constant C such that*

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E} \|B^\varepsilon(t) - \hat{B}^\varepsilon(t)\|^{2p} &\leq C \frac{\delta^{p+1}}{\varepsilon}, \\ \mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \hat{A}^\varepsilon(t)\|^{2p} &\leq C \delta^p + C \frac{\delta^{p+1}}{\varepsilon}, \end{aligned}$$

where C is only dependent of p, T, A_0, B_0 .

Proof. • We prove the first inequality.

Indeed, we have

$$\begin{cases} dB^\varepsilon = \frac{1}{\varepsilon} [\mathcal{L}(B^\varepsilon) + \mathcal{F}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2 dW_2 \\ d\hat{B}^\varepsilon = \frac{1}{\varepsilon} [\mathcal{L}(\hat{B}^\varepsilon) + \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon(k\delta), \hat{B}^\varepsilon)] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2 dW_2, \end{cases}$$

it is easy to see that $B^\varepsilon(t) - \hat{B}^\varepsilon(t)$ satisfies the following SPDE

$$\begin{cases} d(B^\varepsilon - \hat{B}^\varepsilon) = \frac{1}{\varepsilon} [\mathcal{L}(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) \\ \quad \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta)] dt & \text{in } I \times (k\delta, \min\{(k+1)\delta, T\}) \\ (B^\varepsilon - \hat{B}^\varepsilon)(0, t) = 0 = (B^\varepsilon - \hat{B}^\varepsilon)(1, t) & \text{in } (k\delta, \min\{(k+1)\delta, T\}) \\ (B^\varepsilon - \hat{B}^\varepsilon)(x, k\delta) = 0 & \text{in } I. \end{cases} \quad (5.3)$$

For $t \in [0, T]$ with $t \in [k\delta, (k+1)\delta)$, applying Itô formula to (5.3)

$$\begin{aligned}
& \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
&= 2 \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), \frac{1}{\varepsilon} [\mathcal{L}(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) \right. \\
&\quad \left. + \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta)] \right) ds \\
&= \frac{2}{\varepsilon} \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), \mathcal{L}(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) \right. \\
&\quad \left. + \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta) \right) ds \\
&= -\frac{2\beta}{\varepsilon} \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \| (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 ds \\
&\quad + \frac{2}{\varepsilon} \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta) \right) ds.
\end{aligned}$$

By taking mathematical expectation from both sides of above equation, we have

$$\begin{aligned}
& \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
&= -\frac{2\beta}{\varepsilon} \mathbb{E} \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \| (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 ds \\
&\quad + \frac{2}{\varepsilon} \mathbb{E} \int_{k\delta}^t \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta) \right) ds,
\end{aligned}$$

thus, we have

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
&= -\frac{2\beta}{\varepsilon} \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p-2} \| (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 \\
&\quad + \frac{2}{\varepsilon} \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + \eta B^\varepsilon - \eta \hat{B}^\varepsilon + i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta) \right).
\end{aligned}$$

It follows from Lemma 2.1, we have

$$\begin{aligned}
& (B^\varepsilon - \hat{B}^\varepsilon, \mathcal{F}(B^\varepsilon) - \mathcal{F}(\hat{B}^\varepsilon)) \leq 0, \\
& (B^\varepsilon - \hat{B}^\varepsilon, \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon)) \leq 0,
\end{aligned}$$

thus, it holds that

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
&\leq -\frac{2\beta}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} \| (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 + \frac{2\eta}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} \| B^\varepsilon - \hat{B}^\varepsilon \|^2 \\
&\quad + \frac{2}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} \left((B^\varepsilon - \hat{B}^\varepsilon), i\kappa A^\varepsilon - i\kappa A^\varepsilon(k\delta) \right) \\
&\leq -\frac{2\beta\lambda}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{2\eta}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} \\
&\quad + \frac{C}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} \| B^\varepsilon - \hat{B}^\varepsilon \| \| A^\varepsilon - A^\varepsilon(k\delta) \|.
\end{aligned}$$

It follows from the Young inequality (see Lemma 2.2) that

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
&\leq -\frac{2\beta\lambda}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{4\eta}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{C(p,\eta)}{\varepsilon} \mathbb{E} \| A^\varepsilon - A^\varepsilon(k\delta) \|^{2p} \\
&= -\frac{2\beta\lambda-4\eta}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{C(p,\eta)}{\varepsilon} \mathbb{E} \| A^\varepsilon - A^\varepsilon(k\delta) \|^{2p} \\
&= -\frac{4\alpha}{\varepsilon} \mathbb{E} \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{C}{\varepsilon} \mathbb{E} \| A^\varepsilon - A^\varepsilon(k\delta) \|^{2p},
\end{aligned}$$

due to Lemma 5.1, it holds that

$$\frac{d}{dt} \mathbb{E} \|(B^\varepsilon - \hat{B}^\varepsilon)(t)\|^{2p} \leq -\frac{4\alpha}{\varepsilon} \mathbb{E} \|B^\varepsilon - \hat{B}^\varepsilon\|^{2p} + \frac{C}{\varepsilon} \delta^p.$$

Hence, by applying Lemma 2.3 with $\mathbb{E} \|(B^\varepsilon - \hat{B}^\varepsilon)(t)\|^{2p}$, we have

$$\begin{aligned} & \mathbb{E} \|(B^\varepsilon - \hat{B}^\varepsilon)(t)\|^{2p} \\ & \leq \int_{k\delta}^t e^{-\frac{4\alpha}{\varepsilon}(t-s)} \frac{C}{\varepsilon} \delta^p ds \\ & = \frac{C}{\varepsilon} \delta^p \int_{k\delta}^t e^{-\frac{4\alpha}{\varepsilon}(t-s)} ds \\ & = C \frac{\delta^{p+1}}{\varepsilon}. \end{aligned}$$

• We prove the second inequality.

Indeed, we have

$$\begin{cases} dA^\varepsilon = [\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + \mathcal{G}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1 dW_1 \\ d\hat{A}^\varepsilon = [\mathcal{L}(\hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon(t(\delta))) + \mathcal{G}(A^\varepsilon(t(\delta))) + f(A^\varepsilon(t(\delta)), \hat{B}^\varepsilon)]ds + \sigma_1 dW_1, \end{cases}$$

it is easy to see that $A^\varepsilon(t) - \hat{A}^\varepsilon(t)$ satisfies the following SPDE

$$\begin{cases} d(A^\varepsilon - \hat{A}^\varepsilon) = [\mathcal{L}(A^\varepsilon - \hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(t(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(t(\delta))) \\ \quad \eta A^\varepsilon - \eta A^\varepsilon(t(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon]dt & \text{in } I \times (0, T) \\ (A^\varepsilon - \hat{A}^\varepsilon)(0, t) = 0 = (A^\varepsilon - \hat{A}^\varepsilon)(1, t) & \text{in } (0, T) \\ (A^\varepsilon - \hat{A}^\varepsilon)(x, 0) = 0 & \text{in } I. \end{cases} \quad (5.4)$$

For $t \in [0, T]$, applying Itô formula to (5.4)

$$\begin{aligned} & \|(A^\varepsilon - \hat{A}^\varepsilon)(t)\|^{2p} \\ & = 2p \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p-2} ((A^\varepsilon - \hat{A}^\varepsilon), \mathcal{L}(A^\varepsilon - \hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) \\ & \quad + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon) ds \\ & = -2p\beta \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p-2} \|(A^\varepsilon - \hat{A}^\varepsilon)_x\|^2 \\ & \quad + 2p \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p-2} ((A^\varepsilon - \hat{A}^\varepsilon), \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) \\ & \quad + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon) ds \\ & \leq -2p\beta\lambda \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p} ds \\ & \quad + 2p \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p-2} \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\| \|\mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) \\ & \quad + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon\| ds \\ & \leq -2p\beta\lambda \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p} ds \\ & \quad + 2p \int_0^t \|(A^\varepsilon - \hat{A}^\varepsilon)(s)\|^{2p-1} \|\mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) \\ & \quad + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon\| ds. \end{aligned}$$

It follows from the Young inequality (see Lemma 2.2) that

$$\begin{aligned}
& \| (A^\varepsilon - \hat{A}^\varepsilon)(t) \|^{2p} \\
& \leq -2p\beta\lambda \int_0^t \| (A^\varepsilon - \hat{A}^\varepsilon)(s) \|^{2p} ds + p\beta\lambda \int_0^t \| (A^\varepsilon - \hat{A}^\varepsilon)(s) \|^{2p} ds \\
& + C(p, \beta, \lambda) \int_0^t \| \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon \|^{2p} ds \\
& = -p\beta\lambda \int_0^t \| (A^\varepsilon - \hat{A}^\varepsilon)(s) \|^{2p} ds \\
& + C(p, \beta, \lambda) \int_0^t \| \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon \|^{2p} ds \\
& \leq C(p, \beta, \lambda) \int_0^t \| \mathcal{F}(A^\varepsilon) - \mathcal{F}(A^\varepsilon(s(\delta))) + \mathcal{G}(A^\varepsilon) - \mathcal{G}(A^\varepsilon(s(\delta))) + \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) + i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon \|^{2p} ds.
\end{aligned}$$

It follows from Lemma 2.4, Proposition 5.1 and Proposition 4.1 that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \| \mathcal{F}(A^\varepsilon(s)) - \mathcal{F}(A^\varepsilon(s(\delta))) \|^{2p} ds \\
& \leq C \mathbb{E} \int_0^T [\| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \| (\| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2)]^{2p} ds \\
& = C \mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{2p} (\| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2)^{2p} ds \\
& \leq C (\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{4p} ds)^{\frac{1}{2}} \cdot (\mathbb{E} \int_0^T (\| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2)^{4p} ds)^{\frac{1}{2}} \\
& \leq C(p, T) (\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{4p} ds)^{\frac{1}{2}} \\
& \leq C\delta^p,
\end{aligned}$$

by the same method, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \| \mathcal{G}(A^\varepsilon(s)) - \mathcal{G}(A^\varepsilon(s(\delta))) \|^{2p} ds \leq C\delta^p,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \| \eta A^\varepsilon - \eta A^\varepsilon(s(\delta)) \|^{2p} ds \leq C\delta^p.$$

Noting

$$\mathbb{E} \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \leq C \frac{\delta^{p+1}}{\varepsilon},$$

we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \| i\kappa B^\varepsilon - i\kappa \hat{B}^\varepsilon \|^{2p} ds \leq C \frac{\delta^{p+1}}{\varepsilon}.$$

Thus,

$$\mathbb{E} \sup_{0 \leq t \leq T} \| (A^\varepsilon - \hat{A}^\varepsilon)(t) \|^{2p} \leq C\delta^p + C \frac{\delta^{p+1}}{\varepsilon}.$$

□

5.5 The errors of $\hat{A}^\varepsilon - \bar{A}$

Next we prove the strong convergence of the auxiliary process \hat{A}^ε to the averaging solution process \bar{A} .

Lemma 5.2. *There exists a constant $C(T, p)$ such that*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p} \leq C(\varepsilon^{\frac{p}{2}} + \varepsilon^{\frac{2p-1}{4}} + \varepsilon^{\frac{2p-1}{2}} + \varepsilon^{\frac{p-1}{2}} + \varepsilon^{\frac{1}{4}})\varepsilon^{-\frac{1}{8}} + \frac{C}{\sqrt[p]{-\ln \varepsilon}}.$$

Proof. Note that $\hat{A}^\varepsilon, \bar{A}$ satisfy

$$\begin{cases} d\hat{A}^\varepsilon = [\mathcal{L}(\hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon(t(\delta))) + \mathcal{G}(A^\varepsilon(t(\delta))) + f(A^\varepsilon(t(\delta)), \hat{B}^\varepsilon)]dt + \sigma_1 dW_1 \\ d\bar{A} = [\mathcal{L}(\bar{A}) + \mathcal{F}(\bar{A}) + \mathcal{G}(\bar{A}) + \bar{f}(\bar{A})]dt + \sigma_1 dW_1. \end{cases}$$

In mild sense, we introduce the following decomposition

$$\begin{aligned} & \hat{A}^\varepsilon(t) - \bar{A}(t) \\ &= \int_0^t S(t-s)[\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(\bar{A}(s)) + \mathcal{G}(A^\varepsilon(s(\delta))) - \mathcal{G}(\bar{A}(s)) + f(A^\varepsilon(s(\delta)), \hat{B}^\varepsilon(s)) - \bar{f}(\bar{A}(s))]ds \\ &= \int_0^t S(t-s)[\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(\bar{A}(s))]ds \\ &\quad + \int_0^t S(t-s)[\mathcal{G}(A^\varepsilon(s(\delta))) - \mathcal{G}(\bar{A}(s))]ds \\ &\quad + \int_0^t S(t-s)[f(A^\varepsilon(s(\delta)), \hat{B}^\varepsilon(s)) - \bar{f}(\bar{A}(s))]ds \\ &\triangleq J_1 + J_2 + J_3. \end{aligned}$$

- For J_1 , we can rewrite J_1 as

$$\begin{aligned} & J_1 \\ &= \int_0^t S(t-s)[\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(\bar{A}(s))]ds \\ &= \int_0^t S(t-s)[\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))]ds \\ &\quad + \int_0^t S(t-s)[\mathcal{F}(A^\varepsilon(s)) - \mathcal{F}(\hat{A}^\varepsilon(s))]ds \\ &\quad + \int_0^t S(t-s)[\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))]ds \\ &\triangleq J_{11} + J_{12} + J_{13}. \end{aligned}$$

★ For J_{11} , by using the Hölder inequality, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{11}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) [\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|S(t-s) [\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))\| ds \right)^{2p} \\
&= \mathbb{E} \left(\int_0^T \|\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))\| ds \right)^{2p} \\
&\leq \mathbb{E} \left[\left(\int_0^T 1^{\frac{2p}{2p-1}} ds \right)^{2p-1} \cdot \int_0^T \|\mathcal{F}(A^\varepsilon(s(\delta))) - \mathcal{F}(A^\varepsilon(s))\|^{2p} ds \right] \\
&\leq C \mathbb{E} \int_0^T [\|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\| (\|A^\varepsilon(s(\delta))\|_{H^1}^2 + \|A^\varepsilon(s)\|_{H^1}^2)]^{2p} ds \\
&= C \mathbb{E} \int_0^T \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\|^{2p} (\|A^\varepsilon(s(\delta))\|_{H^1}^2 + \|A^\varepsilon(s)\|_{H^1}^2)^{2p} ds \\
&\leq C (\mathbb{E} \int_0^T \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\|^{4p} ds)^{\frac{1}{2}} (\mathbb{E} \int_0^T (\|A^\varepsilon(s(\delta))\|_{H^1}^2 + \|A^\varepsilon(s)\|_{H^1}^2)^{4p} ds)^{\frac{1}{2}}.
\end{aligned}$$

It follows from Proposition 4.1 and Proposition 5.1 that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|J_{11}\|^{2p} \leq C \delta^p.$$

★ For J_{12} , by using the Hölder inequality and the same method in dealing with J_{11} , we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{12}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) [\mathcal{F}(A^\varepsilon(s)) - \mathcal{F}(\hat{A}^\varepsilon(s))] ds \right\|^{2p} \\
&\leq C (\mathbb{E} \int_0^T \|A^\varepsilon(s) - \hat{A}^\varepsilon(s)\|^{4p} ds)^{\frac{1}{2}} (\mathbb{E} \int_0^T (\|A^\varepsilon(s)\|_{H^1}^2 + \|\hat{A}^\varepsilon(s)\|_{H^1}^2)^{4p} ds)^{\frac{1}{2}}.
\end{aligned}$$

It follows from Proposition 4.1 and Proposition 5.1 that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|J_{12}\|^{2p} \leq (C \delta^{2p} + \frac{C}{\varepsilon} \delta^{2p+1})^{\frac{1}{2}} \leq C \delta^p + \frac{C}{\sqrt{\varepsilon}} \delta^{p+\frac{1}{2}}.$$

★ For J_{13} , by using the Hölder inequality and the same method in dealing with J_{11} , we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{13}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) [\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))\| ds \right)^{2p} \\
&\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| (\|\hat{A}^\varepsilon(s)\|_{H^1}^2 + \|\bar{A}(s)\|_{H^1}^2) ds \right)^{2p}.
\end{aligned}$$

In order to deal with the above estimate, we will use the skill of stopping times, this is inspired from [12].

We define the stopping time

$$\tau_n^\varepsilon = \inf\{t > 0 : \|\hat{A}^\varepsilon(t)\|_{H^1}^2 + \|\bar{A}(t)\|_{H^1}^2 > n\}$$

for any $n \geq 1$, and $\varepsilon > 0$.

We have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|J_{13}\|^{2p} \\ &= \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left\| \int_0^t S(t-s) [\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))] ds \right\|^{2p} \\ &\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left(\int_0^t \|\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))\| ds \right)^{2p} \\ &= \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\mathcal{F}(\hat{A}^\varepsilon(s)) - \mathcal{F}(\bar{A}(s))\| ds \right)^{2p} \\ &\leq C \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| (\|\hat{A}^\varepsilon(s)\|_{H^1}^2 + \|\bar{A}(s)\|_{H^1}^2) ds \right)^{2p} \\ &\leq C n^{2p} \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| ds \right)^{2p} \\ &\leq C n^{2p} \mathbb{E} \left[\left(\int_0^{T \wedge \tau_n^\varepsilon} 1^{\frac{2p}{2p-1}} ds \right)^{2p-1} \cdot \int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \right] \\ &\leq C n^{2p} \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\ &\leq C n^{2p} \mathbb{E} \int_0^T \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\ &= C n^{2p} \int_0^T \mathbb{E} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\ &\leq C n^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds. \end{aligned}$$

- For J_2 , we can rewrite J_2 as

$$\begin{aligned} & J_2 \\ &= \int_0^t S(t-s) [\mathcal{G}(A^\varepsilon(s(\delta))) - \mathcal{G}(\bar{A}(s))] ds \\ &= \int_0^t S(t-s) [\mathcal{G}(A^\varepsilon(s(\delta))) - \mathcal{G}(A^\varepsilon(s))] ds \\ &\quad + \int_0^t S(t-s) [\mathcal{G}(A^\varepsilon(s)) - \mathcal{G}(\hat{A}^\varepsilon(s))] ds \\ &\quad + \int_0^t S(t-s) [\mathcal{G}(\hat{A}^\varepsilon(s)) - \mathcal{G}(\bar{A}(s))] ds \\ &\triangleq J_{21} + J_{22} + J_{23}. \end{aligned}$$

- ★ For J_{21} , by using the Hölder inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|^{2p} \\ &\leq C (\mathbb{E} \int_0^T \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\|^{4p} ds)^{\frac{1}{2}} (\mathbb{E} \int_0^T (\|A^\varepsilon(s(\delta))\|_{H^1}^4 + \|A^\varepsilon(s)\|_{H^1}^4)^{4p} ds)^{\frac{1}{2}}. \end{aligned}$$

It follows from Proposition 4.1 and Proposition 5.1 that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|^{2p} \leq C\delta^p.$$

★ For J_{22} , by using the Hölder inequality and the same method in dealing with J_{11} , we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \|J_{22}\|^{2p} \\ &= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) [\mathcal{G}(A^\varepsilon(s)) - \mathcal{G}(\hat{A}^\varepsilon(s))] ds \right\|^{2p} \\ &\leq C(\mathbb{E} \int_0^T \|A^\varepsilon(s) - \hat{A}^\varepsilon(s)\|^{4p} ds)^{\frac{1}{2}} (\mathbb{E} \int_0^T (\|A^\varepsilon(s)\|_{H^1}^4 + \|\hat{A}^\varepsilon(s)\|_{H^1}^4)^{4p} ds)^{\frac{1}{2}}. \end{aligned}$$

It follows from Proposition 4.1 and Proposition 5.1 that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|J_{22}\|^{2p} \leq (C\delta^{2p} + \frac{C}{\varepsilon}\delta^{2p+1})^{\frac{1}{2}} \leq C\delta^p + \frac{C}{\sqrt{\varepsilon}}\delta^{p+\frac{1}{2}}.$$

★ For J_{23} , by using the same method in dealing with J_{13} , we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} \|J_{23}\|^{2p} \\ &\leq C\mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| (\|\hat{A}^\varepsilon(s)\|_{H^1}^4 + \|\bar{A}(s)\|_{H^1}^4) ds \right)^{2p}. \end{aligned}$$

It holds that

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|J_{23}\|^{2p} \\ &= \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left\| \int_0^t S(t-s) [\mathcal{G}(\hat{A}^\varepsilon(s)) - \mathcal{G}(\bar{A}(s))] ds \right\|^{2p} \\ &\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left(\int_0^t \|\mathcal{G}(\hat{A}^\varepsilon(s)) - \mathcal{G}(\bar{A}(s))\| ds \right)^{2p} \\ &= \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\mathcal{G}(\hat{A}^\varepsilon(s)) - \mathcal{G}(\bar{A}(s))\| ds \right)^{2p} \\ &\leq C\mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| (\|\hat{A}^\varepsilon(s)\|_{H^1}^4 + \|\bar{A}(s)\|_{H^1}^4) ds \right)^{2p}. \end{aligned}$$

Noting the fact that when $0 \leq t \leq \tau_n^\varepsilon$, it holds that

$$\|\hat{A}^\varepsilon(t)\|_{H^1}^4 + \|\bar{A}(t)\|_{H^1}^4 \leq (\|\hat{A}^\varepsilon(t)\|_{H^1}^2 + \|\bar{A}(t)\|_{H^1}^2)^2 \leq n^2,$$

thus, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|J_{23}\|^{2p} \\
& \leq Cn^{4p} \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\| ds \right)^{2p} \\
& \leq Cn^{4p} \mathbb{E} \left[\left(\int_0^{T \wedge \tau_n^\varepsilon} 1^{\frac{2p}{2p-1}} ds \right)^{2p-1} \cdot \int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \right] \\
& \leq Cn^{4p} \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
& \leq Cn^{4p} \mathbb{E} \int_0^T \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
& = Cn^{4p} \int_0^T \mathbb{E} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
& \leq Cn^{4p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds.
\end{aligned}$$

- For J_3 , we can rewrite J_3 as

$$\begin{aligned}
& J_3 \\
& = \int_0^t S(t-s) [(f(A^\varepsilon(s\delta)), \hat{B}^\varepsilon) - \bar{f}(A^\varepsilon(s))] ds \\
& \quad + \int_0^t S(t-s) [\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))] ds \\
& \quad + \int_0^t S(t-s) [\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))] ds \\
& = \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s) [(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta))] ds \\
& \quad + \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s) [\bar{f}(A^\varepsilon(k\delta)) - \bar{f}(A^\varepsilon(s))] ds \\
& \quad + \int_{m_t\delta}^t S(t-s) [f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))] ds \\
& \quad + \int_0^t S(t-s) [\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))] ds \\
& \quad + \int_0^t S(t-s) [\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))] ds \\
& \triangleq J_{31} + J_{32} + J_{33} + J_{34} + J_{35},
\end{aligned}$$

where $m_t = \lceil \frac{t}{\delta} \rceil$.

★ For J_{32} , due to Lemma 5.1, it concludes that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{32}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s) [\bar{f}(A^\varepsilon(k\delta)) - \bar{f}(A^\varepsilon(s))] ds \right\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s) [\bar{f}(A^\varepsilon(s(\delta))) - \bar{f}(A^\varepsilon(s))] ds \right\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^{m_t\delta} S(t-s) [\bar{f}(A^\varepsilon(s(\delta))) - \bar{f}(A^\varepsilon(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^{m_t\delta} \|S(t-s) [\bar{f}(A^\varepsilon(s(\delta))) - \bar{f}(A^\varepsilon(s))]\| ds \right)^{2p} \\
&\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^{m_t\delta} \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\| ds \right)^{2p} \\
&\leq C \mathbb{E} \left(\int_0^T \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\| ds \right)^{2p} \\
&\leq C \mathbb{E} \left(\int_0^T 1 ds \right)^{2p-1} \cdot \int_0^T \|A^\varepsilon(s(\delta)) - A^\varepsilon(s)\|^{2p} ds \\
&\leq C \delta^p.
\end{aligned}$$

★ For J_{33} , according to Proposition 4.1 and the global Lipschitz property of f and \bar{f} , we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{33}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_{m_t\delta}^t S(t-s) [f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t \|S(t-s) [f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t \|f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))\| ds \right)^{2p} \\
&\leq C \mathbb{E} \sup_{0 \leq t \leq T} \left[\left(\int_{m_t\delta}^t 1 ds \right)^{2p-1} \cdot \left(\int_{m_t\delta}^t \|f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))\|^{2p} ds \right) \right] \\
&\leq C \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t 1 ds \right)^{2p-1} \cdot \sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t \|f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))\|^{2p} ds \right) \right] \\
&= C \sup_{0 \leq t \leq T} (t - m_t\delta)^{2p-1} \cdot \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t \|f(A^\varepsilon(m_t\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(s))\|^{2p} ds \right) \right] \\
&\leq C \delta^{2p-1} \cdot \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\int_{m_t\delta}^t (\|A^\varepsilon(m_t\delta)\|^{2p} + \|\hat{B}^\varepsilon(s)\|^{2p} + \|A^\varepsilon(s)\|^{2p}) ds \right) \right] \\
&\leq C \delta^{2p-1} \cdot \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\int_0^T (\|A^\varepsilon(m_t\delta)\|^{2p} + \|\hat{B}^\varepsilon(s)\|^{2p} + \|A^\varepsilon(s)\|^{2p}) ds \right) \right] \\
&\leq C \delta^{2p-1} \cdot \mathbb{E} \left[\int_0^T (\|A^\varepsilon(m_t\delta)\|^{2p} + \|\hat{B}^\varepsilon(s)\|^{2p} + \|A^\varepsilon(s)\|^{2p}) ds \right] \\
&\leq C \delta^{2p-1}.
\end{aligned}$$

★ For J_{34} , due to Lemma 5.1, it concludes that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{34}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) [\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|S(t-s) [\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \|\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))\| ds \right)^{2p} \\
&\leq \mathbb{E} \left(\int_0^T \|\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))\| ds \right)^{2p} \\
&\leq C \mathbb{E} \int_0^T \|\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))\|^{2p} ds \\
&\leq C \mathbb{E} \int_0^T \|A^\varepsilon(s) - \hat{A}^\varepsilon(s)\|^{2p} ds \\
&\leq C \delta^p + C \frac{\delta^{p+1}}{\varepsilon}.
\end{aligned}$$

★ For J_{35} , it concludes that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|J_{35}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left\| \int_0^t S(t-s) [\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left(\int_0^t \|S(t-s) [\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \left(\int_0^t \|\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))\| ds \right)^{2p} \\
&\leq \mathbb{E} \left(\int_0^{T \wedge \tau_n^\varepsilon} \|\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))\| ds \right)^{2p} \\
&\leq C \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \|\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))\|^{2p} ds \\
&\leq C \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
&\leq C \mathbb{E} \int_0^T \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
&= C \int_0^T \mathbb{E} \|\hat{A}^\varepsilon(s) - \bar{A}(s)\|^{2p} ds \\
&\leq C \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds.
\end{aligned}$$

★ For J_{31} , by a time shift transformation, we have for any fixed p and $t \in [0, \delta)$ that

$$\begin{aligned}
\hat{B}^\varepsilon(t + p\delta) &= B^\varepsilon(p\delta) + \frac{1}{\varepsilon} \int_{p\delta}^{t+p\delta} \mathcal{L}\hat{B}^\varepsilon(s)ds \\
&\quad + \frac{1}{\varepsilon} \int_{p\delta}^{t+p\delta} [\mathcal{F}(\hat{B}^\varepsilon(s)) + \mathcal{G}(\hat{B}^\varepsilon(s)) + \eta\hat{B}^\varepsilon(s) + i\kappa A^\varepsilon(p\delta)]ds \\
&\quad + \frac{1}{\sqrt{\varepsilon}} \int_{p\delta}^{t+p\delta} \sigma_2 dW_2(s) \\
&= B^\varepsilon(p\delta) + \frac{1}{\varepsilon} \int_0^t \mathcal{L}\hat{B}^\varepsilon(s + p\delta)ds \\
&\quad + \frac{1}{\varepsilon} \int_0^t [\mathcal{F}(\hat{B}^\varepsilon(s + p\delta)) + \mathcal{G}(\hat{B}^\varepsilon(s)) + \eta\hat{B}^\varepsilon(s) + i\kappa A^\varepsilon(p\delta)]ds \\
&\quad + \frac{1}{\sqrt{\varepsilon}} \int_0^t \sigma_2 dW_2^*(s),
\end{aligned}$$

where $W_2^*(t)$ is the shift version of $W_2(t)$ and hence they have the same distribution.

Let $\bar{W}(t)$ be a Wiener process defined on the same stochastic basis and independent of $W_1(t)$ and $W_2(t)$. Construct a process $B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(t) \in L^2(I)$ by means of

$$\begin{aligned}
B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{t}{\varepsilon}) &= B^\varepsilon(p\delta) + \int_0^{\frac{t}{\varepsilon}} \mathcal{L}B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(s)ds \\
&\quad + \int_0^t [\mathcal{F}(B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(s)) + \mathcal{G}(B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(s)) + \eta B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(s) + i\kappa A^\varepsilon(p\delta)]ds \\
&\quad + \int_0^{\frac{t}{\varepsilon}} \sigma_2 d\bar{W}(s) \\
&= B^\varepsilon(p\delta) + \frac{1}{\varepsilon} \int_0^t \mathcal{L}B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{s}{\varepsilon})ds \\
&\quad + \frac{1}{\varepsilon} \int_0^t [\mathcal{F}(B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{s}{\varepsilon})) + \mathcal{G}(B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{s}{\varepsilon})) + \eta B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{s}{\varepsilon}) + i\kappa A^\varepsilon(p\delta)]ds \\
&\quad + \frac{1}{\sqrt{\varepsilon}} \int_0^t \sigma_2 d\bar{\bar{W}}(s),
\end{aligned}$$

here $\bar{\bar{W}}(t)$ is the scaled version of $\bar{W}(t)$.

By comparing the above two equations, we see that

$$(A^\varepsilon(p\delta), \hat{B}^\varepsilon(t + p\delta)) \sim (A^\varepsilon(p\delta), B^{A^\varepsilon(p\delta), B^\varepsilon(p\delta)}(\frac{t}{\varepsilon})), \quad t \in [0, \delta), \quad (5.5)$$

where \sim denotes a coincidence in distribution sense. Thus, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{31}\|^2 \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-(k+1)\delta)S((k+1)\delta-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} S(t-(k+1)\delta) \int_{k\delta}^{(k+1)\delta} S((k+1)\delta-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\{ m_t \sum_{k=0}^{m_t-1} \left\| S(t-(k+1)\delta) \int_{k\delta}^{(k+1)\delta} S((k+1)\delta-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \right\} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\{ m_t \sum_{k=0}^{m_t-1} \left\| \int_{k\delta}^{(k+1)\delta} S((k+1)\delta-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \right\} \\
&\leq \left[\frac{T}{\delta} \right] \mathbb{E} \left\{ \sum_{k=0}^{\left[\frac{T}{\delta} \right]-1} \left\| \int_{k\delta}^{(k+1)\delta} S((k+1)\delta-s)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \right\} \\
&= \left[\frac{T}{\delta} \right]^2 \varepsilon^2 \max_{0 \leq k \leq \left[\frac{T}{\delta} \right]-1} \mathbb{E} \left\| \int_0^{\frac{\delta}{\varepsilon}} S(\delta-s\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s\varepsilon+k\delta)) - \bar{f}(A^\varepsilon(k\delta)))ds] \right\|^2 \\
&= 2 \left[\frac{T}{\delta} \right]^2 \varepsilon^2 \max_{0 \leq k \leq \left[\frac{T}{\delta} \right]-1} \int_I \mathbb{E} \left\{ \int_0^{\frac{\delta}{\varepsilon}} \int_r^{\frac{\delta}{\varepsilon}} S(\delta-s\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s\varepsilon+k\delta)) - \bar{f}(A^\varepsilon(k\delta)))] \right. \\
&\quad \cdot S(\delta-r\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(r\varepsilon+k\delta)) - \bar{f}(A^\varepsilon(k\delta)))] ds dr \Big\} dx \\
&\triangleq 2 \left[\frac{T}{\delta} \right]^2 \varepsilon^2 \max_{0 \leq k \leq \left[\frac{T}{\delta} \right]-1} \int_0^{\frac{\delta}{\varepsilon}} \int_r^{\frac{\delta}{\varepsilon}} \mathcal{J}_k(s, r) ds dr,
\end{aligned}$$

where

$$\begin{aligned}
& \mathcal{J}_k(s, r) \\
&= \int_I \mathbb{E} \{ S(\delta-s\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s\varepsilon+k\delta)) - \bar{f}(A^\varepsilon(k\delta)))] \\
&\quad \cdot S(\delta-r\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(r\varepsilon+k\delta)) - \bar{f}(A^\varepsilon(k\delta)))] \} dx.
\end{aligned}$$

It follows from (5.5) and the property of semigroup $\{S(t)\}_{t \geq 0}$ that

$$\begin{aligned}
& \mathcal{J}_k(s, r) \\
&= \mathbb{E} \int_I \{S(\delta - s\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s\varepsilon + k\delta)) - \bar{f}(A^\varepsilon(k\delta))) \\
&\quad \cdot S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(r\varepsilon + k\delta)) - \bar{f}(A^\varepsilon(k\delta)))]\} dx \\
&= \mathbb{E} \int_I \{S(\delta - s\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta))) \\
&\quad \cdot S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)) - \bar{f}(A^\varepsilon(k\delta)))]\} dx \\
&= \mathbb{E} \int_I \{S(\delta - s\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta))) \\
&\quad \cdot \mathbb{E}^{B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)}(S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s-r)) - \bar{f}(A^\varepsilon(k\delta)))]\} dx \\
&\leq \{\mathbb{E} \int_I \{S(\delta - s\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta)))]^2 dx\}^{\frac{1}{2}} \\
&\quad \cdot \{\mathbb{E} \int_I \{\mathbb{E}^{B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)}(S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s-r)) - \bar{f}(A^\varepsilon(k\delta)))]^2 dx\}^{\frac{1}{2}} \\
&= \{\mathbb{E} \|S(\delta - s\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta)))]\|^2\}^{\frac{1}{2}} \\
&\quad \cdot \{\mathbb{E} \|\mathbb{E}^{B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)}(S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s-r)) - \bar{f}(A^\varepsilon(k\delta)))]\|^2\}^{\frac{1}{2}} \\
&= \{\mathbb{E} \|S(\delta - s\varepsilon)[f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta))]\|^2\}^{\frac{1}{2}} \\
&\quad \cdot \{\mathbb{E} \|\mathbb{E}^{B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)}(S(\delta - r\varepsilon)[(f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s-r)) - \bar{f}(A^\varepsilon(k\delta)))]\|^2\}^{\frac{1}{2}} \\
&\leq \{\mathbb{E} \|f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s)) - \bar{f}(A^\varepsilon(k\delta))\|^2\}^{\frac{1}{2}} \\
&\quad \cdot \{\mathbb{E} \|\mathbb{E}^{B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(r)}[f(A^\varepsilon(k\delta), B^{A^\varepsilon(k\delta), B^\varepsilon(k\delta)}(s-r)) - \bar{f}(A^\varepsilon(k\delta))]\|^2\}^{\frac{1}{2}}.
\end{aligned}$$

In view of the above inequality, Lemma 3.1 and the method in [13, 14, 15, 16], there holds

$$\mathcal{J}_k(s, r) \leq C e^{-\alpha(s-r)}.$$

Thus if choose $\delta = \delta(\varepsilon)$ such that $\frac{\delta}{\varepsilon}$ sufficiently large, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq T} \|J_{31}\|^2 \\
&\leq 2\left[\frac{T}{\delta}\right]^2 \varepsilon^2 \max_{0 \leq k \leq \left[\frac{T}{\delta}\right]-1} \int_0^{\frac{\delta}{\varepsilon}} \int_r^{\frac{\delta}{\varepsilon}} \mathcal{J}_k(s, r) ds dr \\
&\leq C\left[\frac{T}{\delta}\right]^2 \varepsilon^2 \max_{0 \leq k \leq \left[\frac{T}{\delta}\right]-1} \int_0^{\frac{\delta}{\varepsilon}} \int_r^{\frac{\delta}{\varepsilon}} e^{-\alpha(s-r)} ds dr \\
&\leq C \frac{\varepsilon}{\delta}.
\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{31}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \sum_{k=0}^{m_t-1} \int_{k\delta}^{(k+1)\delta} S(t-s) [(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))] ds \right\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^{m_t\delta} S(t-s) [(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^{m_t\delta} \|S(t-s) [(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \left(\int_0^T \|S(t-s) [(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))]\| ds \right)^{2p} \\
&\leq \mathbb{E} \left(\int_0^T \|(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))\| ds \right)^{2p} \\
&\leq \mathbb{E} \left[\left(\int_0^T 1 ds \right)^{2p-1} \cdot \int_0^T \|(f(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s)) - \bar{f}(A^\varepsilon(k\delta)))\|^{2p} ds \right] \\
&\leq C(p, T),
\end{aligned}$$

thus,

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \|J_{31}\|^{2p} \\
&\leq (\mathbb{E} \sup_{0 \leq t \leq T} \|J_{31}\|^{2(2p-1)})^{\frac{1}{2}} (\mathbb{E} \sup_{0 \leq t \leq T} \|J_{31}\|^2)^{\frac{1}{2}} \\
&\leq C(p, T) \sqrt{\frac{\varepsilon}{\delta}}.
\end{aligned}$$

With the help of the above estimates, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p} \\
&\leq C\delta^p + C\delta^p + \frac{C}{\sqrt{\varepsilon}}\delta^{p+\frac{1}{2}} + Cn^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds \\
&\quad + C\delta^p + C\delta^p + \frac{C}{\sqrt{\varepsilon}}\delta^{p+\frac{1}{2}} + Cn^{4p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds \\
&\quad + C\delta^p + C\delta^{2p-1} + \sqrt{\frac{\varepsilon}{\delta}} + C\delta^p + C\frac{\delta^{p+1}}{\varepsilon} + C \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds \\
&\leq C(\delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \sqrt{\frac{\varepsilon}{\delta}}) + Cn^{4p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(r) - \bar{A}(r)\|^{2p} ds.
\end{aligned}$$

By using the Gronwall inequality, we have

$$\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_n^\varepsilon} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p} \leq C(\delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \sqrt{\frac{\varepsilon}{\delta}}) e^{Cn^{4p}},$$

this implies that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p} \cdot 1_{\{T \leq \tau_n^\varepsilon\}} \right) \leq C(\delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \sqrt{\frac{\varepsilon}{\delta}}) e^{Cn^{4p}}.$$

On the other hand, due to Proposition 4.1, we have

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p} \cdot 1_{\{T > \tau_n^\varepsilon\}}) \\
& \leq \mathbb{E}(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{4p})^{\frac{1}{2}} \cdot (\mathbb{E}1_{\{T > \tau_n^\varepsilon\}})^{\frac{1}{2}} \\
& \leq \frac{C}{\sqrt{n}}.
\end{aligned}$$

Hence, we have

$$\mathbb{E}(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p}) \leq C(\delta^p + \frac{\delta^{p+\frac{1}{2}}}{\sqrt{\varepsilon}} + \delta^{2p-1} + \frac{\delta^{p+1}}{\varepsilon} + \sqrt{\frac{\varepsilon}{\delta}})e^{Cn^{4p}} + \frac{C}{\sqrt{n}},$$

if we take $n = \sqrt[4p]{-\frac{1}{8C} \ln \varepsilon}$, $\delta = \varepsilon^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p}) \\
& \leq C(\varepsilon^{\frac{p}{2}} + \frac{\varepsilon^{\frac{p}{2} + \frac{1}{4}}}{\sqrt{\varepsilon}} + \varepsilon^{\frac{2p-1}{2}} + \frac{\varepsilon^{\frac{p+1}{2}}}{\varepsilon} + \sqrt{\frac{\varepsilon}{\varepsilon^{\frac{1}{2}}}}\varepsilon^{-\frac{1}{8}} + \frac{C}{\sqrt[8p]{-\frac{1}{8C} \ln \varepsilon}}) \\
& = C(\varepsilon^{\frac{p}{2}} + \varepsilon^{\frac{2p-1}{4}} + \varepsilon^{\frac{2p-1}{2}} + \varepsilon^{\frac{p-1}{2}} + \varepsilon^{\frac{1}{4}})\varepsilon^{-\frac{1}{8}} + C(\frac{1}{-\ln \varepsilon})^{\frac{1}{8p}}.
\end{aligned}$$

□

5.6 Proof of Theorem 1.2

By taking $\delta = \varepsilon^{\frac{1}{2}}$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \hat{A}^\varepsilon(t)\|^{2p} \leq C\varepsilon^{\frac{p}{2}} + C\varepsilon^{\frac{p-1}{2}},$$

if $p > \frac{5}{4}$, we have

$$\mathbb{E}(\sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - \bar{A}(t)\|^{2p}) \leq C(\frac{1}{-\ln \varepsilon})^{\frac{1}{8p}},$$

thus, we have

$$\mathbb{E}(\sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{2p}) \leq C(\frac{1}{-\ln \varepsilon})^{\frac{1}{8p}}.$$

If $0 < p \leq \frac{5}{4}$, for any $\kappa > 0$, it holds that

$$\begin{aligned}
& \mathbb{E}(\sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{2p}) \\
& \leq (\mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{\frac{5}{2} + \kappa})^{\frac{2p}{\frac{5}{2} + \kappa}} (\mathbb{E}1)^{1 - \frac{2p}{\frac{5}{2} + \kappa}} \\
& = C(\kappa)(\mathbb{E} \sup_{0 \leq t \leq T} \|A^\varepsilon(t) - \bar{A}(t)\|^{\frac{5}{2} + \kappa})^{\frac{2p}{\frac{5}{2} + \kappa}} \\
& \leq C(\kappa)[(\frac{1}{-\ln \varepsilon})^{\frac{1}{4(\frac{5}{2} + \kappa)}}]^{\frac{2p}{\frac{5}{2} + \kappa}} \\
& = C(\kappa)(\frac{1}{-\ln \varepsilon})^{\frac{2p}{(5+2\kappa)^2}}.
\end{aligned}$$

This completes the proof of Theorem 1.2.

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